SLOW INVARIANT MANIFOLDS AS CURVATURE OF THE FLOW OF DYNAMICAL SYSTEMS

Jean-Marc Ginoux, Bruno Rossetto, Leon Chua

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JEAN-MARC GINOUX, BRUNO ROSSETTO, LEON O. CHUA[†]
Laboratoire PROTEE,
I.U.T. of Toulon, Université du Sud,
B.P. 20132, 83957, LA GARDE Cedex, France
[†]*EECS Department University of California, Berkeley*
253 Cory Hall #1770, Berkeley, CA 94720-1770
ginoux@univ-tln.fr, rossetto@univ-tln.fr chua@eecs.berkeley.edu

website: http://ginoux.univ-tln.fr, http://rossetto.univ-tln.fr,
http://www.eecs.berkeley.edu/∼chua

Abstract

Considering trajectory curves, integral of n-dimensional dynamical systems, within the framework of Differential Geometry as curves in Euclidean n-space it will be established in this article that the curvature of the flow, i.e., the curvature of the trajectory curves of any n-dimensional dynamical system directly provides its slow manifold analytical equation the invariance of which will be then proved according to Darboux theory. Thus, it will be stated that the flow curvature method, which uses neither eigenvectors nor asymptotic expansions but only involves time derivatives of the velocity vector field, constitutes a general method simplifying and improving the slow invariant manifold analytical equation determination of high-dimensional dynamical systems. Moreover, it will be shown that this method generalizes the Tangent Linear System Approximation and encompasses the so-called Geometric Singular Perturbation Theory. Then, slow invariant manifolds analytical equation of paradigmatic Chua's piecewise linear and cubic models of dimensions three, four, and five will be provided as tutorial examples exemplifying this method as well as those of high-dimensional dynamical systems.

Keywords: differential geometry; curvature; torsion; Gram-Schmidt algorithm; Darboux invariant.

1 Introduction

Dynamical systems consisting of *nonlinear* differential equations are generally not integrable. In his famous memoirs: *Sur les courbes définies par une équation différentielle*, Poincaré [1881-1886] faced to this problem proposed to study *trajectory curves* properties in the *phase space*.

"...any differential equation can be written as:

$$\frac{dx_1}{dt} = X_1, \quad \frac{dx_2}{dt} = X_2, \quad \dots, \quad \frac{dx_n}{dt} = X_n$$

where X are integer polynomials.

If t is considered as the time, these equations will define the motion of a variable point in a space of dimension n."

– Poincaré [1885, p. 168] –

Let's consider the following system of differential equations defined in a compact E included in $\mathbb{R}$ as:

$$\frac{d\vec{X}}{dt} = \overrightarrow{\Im}\left(\vec{X}\right) \tag{1}$$

with

$$\vec{X} = [x_1, x_2, ..., x_n]^t \in E \subset \mathbb{R}^n$$

and

$$\overrightarrow{\Im}\left(\vec{X}\right) = \left[f_1\left(\vec{X}\right), f_2\left(\vec{X}\right), ..., f_n\left(\vec{X}\right)\right]^t \in E \subset \mathbb{R}^n$$

The vector $\overrightarrow{\Im}\left(\vec{X}\right)$ defines a velocity vector field in E whose components f_i which are supposed to be continuous and infinitely differentiable with respect to all x_i and t, i.e., are C^∞ functions in E and with values included in $\mathbb{R}$, satisfy the assumptions of the Cauchy-Lipschitz theorem. For more details, see for example Coddington *et al.* [1955]. A solution of this system is a *trajectory curve* $\vec{X}(t)$ tangent[1] to $\overrightarrow{\Im}$ whose values define the *states* of the *dynamical system* described by the Eq. (1). Since none of the components f_i of the velocity vector field depends here explicitly on time, the system is said to be *autonomous*.

Thus, *trajectory curves* integral of dynamical systems (1) regarded as n-dimensional *curves*, possess local metrics properties, namely *curvatures* which can be analytically[2] deduced from the so-called Frénet formulas recalled in the next section. For low dimensions two and three the concept of *curvatures* may

[1] Except at the *fixed points*.

[2] Since only time derivatives of the *trajectory curves* are involved in the *curvature* formulas.

be simply exemplified. A three-dimensional[3] curve for example has two *curvatures: curvature* and *torsion* which are also known as *first* and *second curvature*. *Curvature*[4] measures, so to speak, the deviation of the curve from a straight line in the neighbourhood of any of its points. While the *torsion*[5] measures, roughly speaking, the magnitude and sense of deviation of the curve from the *osculating plane*[6] in the neighbourhood of the corresponding point of the curve, or, in other words, the rate of change of the *osculating plane*. Physically, a three-dimensional curve may be obtained from a straight line by bending (*curvature*) and twisting (torsion). For high dimensions greater than three, say n, a n-dimensional curve has $(n-1)$ *curvatures* which may be computed while using the Gram-Schmidt orthogonalization process [Gluck, 1966]. This procedure, presented in Appendix, also enables to define the Frénet formulas for a n-dimensional curve.

In a recent publication [Ginoux *et al.*, 2006] it has been established that the location of the point where the *curvature of the flow*, i.e., the *curvature* of the *trajectory curves* integral of any *slow-fast dynamical systems* of low dimensions two and three vanishes directly provides the *slow invariant manifold* analytical equation associated to such dynamical systems. So, in this work the new approach proposed by Ginoux *et al.* [2006] is generalized to high-dimensional dynamical systems.

The main result of this work presented in the first section establishes that *curvature of the flow*, i.e., *curvature* of *trajectory curves* of any n-dimensional dynamical system directly provides its *slow manifold* analytical equation the *invariance* of which is proved according to *Darboux Theorem*.

Then, Chua's piecewise linear models of dimensions three, four and five are used in the second section to exemplify this result. Indeed it has been already established [Chua 1986] that such *slow-fast dynamical systems* exhibit *trajectory curves* in the shape of *scrolls* lying on *hyperplanes* the equations of which are well-know. So, it is possible to analytically compute these *hyperplanes* equations while using the *curvature of the flow* and then the comparison leads to a total identity between both equations. Moreover, it is established in the case of piecewise linear models that such *hyperplanes* are no more than *osculating hyperplanes* the invariance of which is stated according to *Darboux Theorem*. Then, *slow invariant manifolds* analytical equations of nonlinear high-dimensional dynamical systems such as fourth-order and fifth-order cubic Chua's circuit [Liu *et al.*, 2007, Hao *et al.*, 2005]and fifth-order magnetoconvection system [Knobloch et al., 1981] are directly provided while using the *curvature of the flow* and *Darboux Theorem*.

In the discussion, a comparison with various methods of *slow invariant manifold* analytical equation determination such as *Tangent Linear System Approximation* [Rossetto *et al.*, 1998] and *Geometric Singular Perturbation Theory* [Fenichel, 1979] highlights that, since it uses neither eigenvectors nor asymp-

[3] A two-dimensional curve, i.e., a plane *curve* has a *torsion* vanishing identically.

[4] The notion of *curvature* of a plane curve first appears in the work of Apollonius of Perga.

[5] The name *torsion* is due to L.I. Vallée, *Traité de Géométrie Descriptive*.

[6] The *osculating plane* is defined as the plane spanned by the instantaneous velocity and acceleration vectors.

totic expansions but simply involves time derivatives of the velocity vector field, *curvature of the flow* constitutes a general method simplifying and improving the *slow invariant manifold* analytical equation determination of any high-dimensional dynamical systems.

In the appendix, definitions inherent to *Differential Geometry* such as the concept of n-dimensional *smooth curves*, *generalized Frénet frame* and *curvatures* definitions are briefly recalled as well as the Gram-Schmidt orthogonalization process for computing *curvatures* of *trajectory curves* in Euclidean n-space. Then, it is shown that the *flow curvature method* generalizes the *Tangent Linear System Approximation* [Rossetto *et al.*, 1998] and encompasses the so-called *Geometric Singular Perturbation Theory* [Fenichel, 1979].

2 Slow invariant manifold analytical equation

The concept of *invariant manifolds* plays a very important role in the stability and structure of dynamical systems and especially for *slow-fast dynamical systems* or *singularly perturbed systems*. Since the beginning of the twentieth century it has been subject to a wide range of seminal research. The classical geometric theory developed originally by Andronov [1937], Tikhonov [1948] and Levinson [1949] stated that *singularly perturbed systems* possess *invariant manifolds* on which trajectories evolve slowly and toward which nearby orbits contract exponentially in time (either forward and backward) in the normal directions. These manifolds have been called asymptotically stable (or unstable) *slow manifolds*. Then, Fenichel [1971-1979] theory[7] for the persistence of normally hyperbolic invariant manifolds enabled to establish the local invariance of *slow manifolds* that possess both expanding and contracting directions and which were labeled *slow invariant manifolds*.

Thus, various methods have been developed in order to determine the *slow invariant manifold* analytical equation associated to *singularly perturbed systems*. The essential works of Wasow [1965], Cole [1968], O'Malley [1974, 1991] and Fenichel [1971-1979] to name but a few, gave rise to the so-called *Geometric Singular Perturbation Theory* and the problem for finding the *slow invariant manifold* analytical equation turned into a regular perturbation problem in which one generally expected, according to O'Malley [1974 p. 78, 1991 p. 21] the asymptotic validity of such expansion to breakdown. Another method called: *tangent linear system approximation*, developed by Rossetto *et al.* [1998], consisted in using the presence of a "fast" eigenvalue in the functional jacobian matrix of low-dimensional (2 and 3) dynamical systems. Within the framework of application of the Tikhonov's theorem [1952], this method used the fact that in the vicinity of the *slow manifold* the eigenmode associated with the "fast" eigenvalue was evanescent. Thus, the *tangent linear system approximation* method provided the *slow manifold* analytical equation of low-dimensional dynamical systems according to the "slow" eigenvectors of the *tangent linear system*, i.e., according to the "slow" eigenvalues. Nevertheless, the presence of

[7] independently developed in Hirsch *et al.*, [1977]

these eigenvalues (real or complex conjugated) prevented from expressing this equation explicitly. Also to solve this problem it was necessary to make such equation independent of the "slow" eigenvalues. This could be carried out by multiplying it by "conjugated" equations leading to a *slow manifold* analytical equation independent of the "slow" eigenvalues of the *tangent linear system*. Then, it was established in [Ginoux *et al.*, 2006] that the resulting equation was identically corresponding in dimension two to the *curvature* (*first curvature*) of the flow and in dimension three to the *torsion* (*second curvature*).

So, in this work the new approach proposed by Ginoux *et al.* [2006] is generalized to high-dimensional dynamical systems. Thus, the main result of this work established in the next section is that *curvature of the flow*, i.e., *curvature* of *trajectory curves* of any n-dimensional dynamical system directly provides its *slow manifold* analytical equation the *invariance* of which is established according to *Darboux Theorem*. Since it uses neither eigenvectors nor asymptotic expansions but simply involves time derivatives of the velocity vector field, it constitutes a general method simplifying and improving the *slow invariant manifold* analytical equation determination of high-dimensional dynamical systems.

2.1 Slow manifold of high-dimensional dynamical systems

In the framework of *Differential Geometry*[8], *trajectory curves* $\vec{X}(t)$ integral of n-dimensional dynamical systems (1) satisfying the assumptions of the Cauchy-Lipschitz theorem may be regarded as n-dimensional *smooth curves*, i.e., *smooth curves* in Euclidean $n-$space *parametrized in terms of time*.

Proposition 2.1. *The location of the points where the curvature of the flow, i.e., the curvature of the trajectory curves of any n-dimensional dynamical system vanishes directly provides its $(n-1)$-dimensional slow invariant manifold analytical equation which reads:*

$$\phi\left(\vec{X}\right) = \dot{\vec{X}} \cdot \left(\ddot{\vec{X}} \wedge \dddot{\vec{X}} \wedge \ldots \wedge \overset{(n)}{\vec{X}}\right) = det\left(\dot{\vec{X}}, \ddot{\vec{X}}, \dddot{\vec{X}}, \ldots, \overset{(n)}{\vec{X}}\right) = 0 \qquad (2)$$

where $\overset{(n)}{\vec{X}}$ *represents the time derivatives of* $\vec{X} = [x_1, x_2, ..., x_n]^t$.

Proof. Let's notice that *inner product* (2) reads:

$$\dot{\vec{X}} \cdot \left(\ddot{\vec{X}} \wedge \dddot{\vec{X}} \wedge \ldots \wedge \overset{(n)}{\vec{X}}\right) = \left[\dot{\vec{X}}, \ddot{\vec{X}}, \ldots, \overset{(n)}{\vec{X}}\right]$$

where $\wedge$ represents the *wedge product*. Let's consider the vectors $\vec{u}_1(t)$, $\vec{u}_2(t)$, ..., $\vec{u}_i(t)$ forming an orthogonal basis defined with the Gram-Schmidt process [Lichnerowicz, 1950 p. 30, Gluck, 1966]. Then, while using identity (54) established in appendix

[8] See appendix for defintions

$$\left[\dot{\vec{X}}, \ddot{\vec{X}}, \ldots, \overset{(n)}{\vec{X}}\right] = \|\vec{u}_1\| \|\vec{u}_2\| \ldots \|\vec{u}_n\| \tag{3}$$

curvature (51) may be written:

$$\kappa_i = \frac{\|\vec{u}_{i+1}(t)\|}{\|\vec{u}_1(t)\| \|\vec{u}_i(t)\|} = \frac{\left[\dot{\vec{X}}, \ddot{\vec{X}}, \ldots, \overset{(i+1)}{\vec{X}}\right]}{\|\vec{u}_1\|^2 \|\vec{u}_2\| \ldots \|\vec{u}_{i-1}\| \|\vec{u}_i\|^2} \tag{4}$$

First and second *curvatures* of space *curves*, i.e., *curvature* (52) and *torsion* (53) may be, for example, found again. Thus, for $i = 1$ identity (54) provides: $\left[\dot{\vec{X}}, \ddot{\vec{X}}\right] = \|\vec{u}_1\| \|\vec{u}_2\|$ and *curvature* κ_1 reads:

$$\kappa_1 = \frac{\|\vec{u}_2\|}{\|\vec{u}_1\| \|\vec{u}_1\|} = \frac{\left[\dot{\vec{X}}, \ddot{\vec{X}}\right]}{\|\vec{u}_1\|^3} = \frac{\left\|\dot{\vec{X}} \wedge \ddot{\vec{X}}\right\|}{\left\|\dot{\vec{X}}\right\|^3} = \frac{\left\|\vec{\gamma} \wedge \overrightarrow{V}\right\|}{\left\|\overrightarrow{V}\right\|^3}$$

For $i = 2$, while using identity (54): $\left[\dot{\vec{X}}, \ddot{\vec{X}}, \dddot{\vec{X}}\right] = \|\vec{u}_1\| \|\vec{u}_2\| \|\vec{u}_3\|$, the Gram-Schmidt orthogonalization process (50) for the expression of vectors $\vec{u}_1(t)$ and $\vec{u}_2(t)$ and the Lagrange identity $\|\vec{u}_1\|^2 \|\vec{u}_2\|^2 = \left\|\dot{\vec{X}} \wedge \ddot{\vec{X}}\right\|^2$ *torsion* κ_2 reads:

$$\kappa_2 = \frac{\|\vec{u}_3(t)\|}{\|\vec{u}_1(t)\| \|\vec{u}_2(t)\|} = \frac{\left[\dot{\vec{X}}, \ddot{\vec{X}}, \dddot{\vec{X}}\right]}{\|\vec{u}_1\|^2 \|\vec{u}_2\|^2} = \frac{\dot{\vec{X}} \cdot \left(\ddot{\vec{X}} \wedge \dddot{\vec{X}}\right)}{\left\|\dot{\vec{X}} \wedge \ddot{\vec{X}}\right\|^2} = -\frac{\dot{\vec{\gamma}} \cdot \left(\vec{\gamma} \wedge \overrightarrow{V}\right)}{\left\|\vec{\gamma} \wedge \overrightarrow{V}\right\|^2}$$

Thus, the location of the point where the *curvature of the flow* (4) vanishes, i.e., the location of the point where the *inner product* vanishes defines a $(n-1)$-*dimensional* manifold associated to any n-dimensional dynamical system (1):

$$\begin{aligned} \phi\left(\vec{X}\right) &= \left[\vec{X}, \dot{\vec{X}}, \ddot{\vec{X}}, \ldots, \overset{(n)}{\vec{X}}\right] \\ &= \dot{\vec{X}} \cdot \left(\ddot{\vec{X}} \wedge \dddot{\vec{X}} \wedge \ldots \wedge \overset{(n)}{\vec{X}}\right) \\ &= det\left(\dot{\vec{X}}, \ddot{\vec{X}}, \dddot{\vec{X}}, \ldots, \overset{(n)}{\vec{X}}\right) = 0 \end{aligned} \tag{5}$$

□

The invariance of such manifold is then established while using the *Darboux Theorem* presented below.

2.2 Darboux invariance theorem

According to Schlomiuk [1993] and Llibre *et al.* [2007] it seems that in his memoir entitled: *Sur les équations différentielles algébriques du premier ordre et du premier degré,* Gaston Darboux [1878, p. 71, 1878$_c$, p.1012] has been the first to define the concept of *invariant manifold.* Let's consider a n-dimensional dynamical system (1) describing "the motion of a variable point in a space of dimension n." Let $\vec{X} = [x_1, x_2, \ldots, x_n]^t$ be the coordinates of this point and $\overrightarrow{V} = [\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n]^t$ the corresponding velocity vector.

Proposition 2.2. *The manifold defined by* $\phi\left(\vec{X}\right) = 0$ *where* ϕ *is a* C^1 *in an open set* U *is invariant with respect to the flow of (1) if there exists a* C^1 *function denoted* $K\left(\vec{X}\right)$ *and called cofactor which satisfies:*

$$L_{\overrightarrow{V}}\phi\left(\vec{X}\right) = K\left(\vec{X}\right)\phi\left(\vec{X}\right) \tag{6}$$

for all $\vec{X} \in U$ *and with the Lie derivative operator defined as:*

$$L_{\overrightarrow{V}}\phi = \overrightarrow{V} \cdot \overrightarrow{\nabla}\phi = \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i}\dot{x}_i = \frac{d\phi}{dt}.$$

In the following invariance of the slow manifold will be established according to what will be referred as Darboux Theorem.

Proof. Lie derivative of the *inner product* (2) reads:

$$L_{\overrightarrow{V}}\phi\left(\vec{X}\right) = \dot{\vec{X}} \cdot \left(\ddot{\vec{X}} \wedge \dddot{\vec{X}} \wedge \ldots \wedge \overset{(n+1)}{\vec{X}}\right) = \left[\dot{\vec{X}}, \ddot{\vec{X}}, \dddot{\vec{X}}, \ldots, \overset{(n+1)}{\vec{X}}\right] \tag{7}$$

Moreover, starting from the identity $\ddot{\vec{X}} = J\dot{\vec{X}}$ where J is the functional jacobian matrix associated to any n-dimensional dynamical system (1) it can be established that:

$$\overset{(n+1)}{\vec{X}} = J^n\dot{\vec{X}} \qquad \text{if} \qquad \frac{dJ}{dt} = 0$$

where J^n represents the n^{th}power of J.

As an example, $\ddot{\vec{X}} = J\dot{\vec{X}} \quad \Leftrightarrow \quad \vec{\gamma} = J\vec{V}$. Then, it follows that

$$\overset{(n+1)}{\vec{X}} = JJ^{n-1}\dot{\vec{X}} = J\overset{(n)}{\vec{X}} \tag{8}$$

Replacing $\overset{(n+1)}{\vec{X}}$ in expression (7) by Eq. (8) we have:

$$L_{\overrightarrow{V}}\phi\left(\vec{X}\right) = \dot{\vec{X}} \cdot \left(\ddot{\vec{X}} \wedge \dddot{\vec{X}} \wedge \ldots \wedge J \overset{(n)}{\vec{X}}\right) = \left[\dot{\vec{X}}, \ddot{\vec{X}}, \dddot{\vec{X}}, \ldots, J \overset{(n)}{\vec{X}}\right] \tag{9}$$

Then, identity (61) established in appendix leads to:

$$L_{\overrightarrow{V}}\phi\left(\vec{X}\right) = Tr\left[J\right] \dot{\vec{X}} \cdot \left(\ddot{\vec{X}} \wedge \dddot{\vec{X}} \wedge \ldots \wedge \overset{(n)}{\vec{X}}\right) = Tr\left[J\right] \phi\left(\vec{X}\right) = K\left(\vec{X}\right)\phi\left(\vec{X}\right)$$

where $K\left(\vec{X}\right) = Tr\left[J\right]$ represents the trace of the functional jacobian matrix.

So, according to *Darboux Theorem* invariance of the *slow manifold* analytical equation of any n-dimensional dynamical system is established provided that the functional jacobian matrix is stationary (7).

□

Note. Since the *slow invariant manifold* analytical equation (2) is defined starting from the velocity vector field all fixed points are belonging to it.

3 Chua's piecewise linear models

It has been established that Chua's piecewise linear models exhibit *trajectory curves* in the shape of *double scrolls* lying on *hyperplanes* the equations of which have been already analytically computed [Chua *et al.*, 1986, Rossetto 1993, Liu *et al.*, 2007]. The aim of this section is first to provide these *hyperplanes* equations with a classical method and then with the new one proposed, i.e., with *curvature of the flow*. A comparison of *hyperplanes* equations given by both methods leads to a total identity. Then, it is stated according to *Darboux Theorem* [1878] that these *hyperplanes* are overflowing invariant with respect to the flow of Chua's models and are, consequently, *invariant manifolds*. Moreover, it is also established, in the framework of the *Differential Geometry*, that such *hyperplanes* are no more than "*osculating hyperplanes".*

3.1 Three-dimensional Chua's system

The piecewise linear Chua's circuit [Chua *et al.*, 1986] is an electronic circuit comprising an inductance L_1, an active resistor R, two capacitors C_1 and C_2, and a nonlinear resistor. Chua's circuit can be accurately modeled by means of a system of three coupled first-order ordinary differential equations in the variables $x_1(t)$, $x_2(t)$ and $x_3(t)$, which give the voltages in the capacitors C_1 and C_2, and the intensity of the electrical current in the inductance L_1, respectively. These equations called *global unfolding* of Chua's circuit are written in a dimensionless form:

$$
\overrightarrow{V}\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{pmatrix} = \overrightarrow{\Im}\begin{pmatrix} f_1(x_1,x_3,x_3) \\ f_2(x_1,x_3,x_3) \\ f_3(x_1,x_3,x_3) \end{pmatrix} = \begin{pmatrix} \alpha(x_2-x_1-k(x_1)) \\ x_1-x_2+x_3 \\ -\beta x_2 \end{pmatrix} \tag{10}
$$

The function $k(x_1)$ describes the electrical response of the nonlinear resistor, i.e., its characteristics which is a piecewise linear function defined by:

$$
k(x_1) = \begin{cases} bx_1 + a - b \ x_1 \geqslant 1 \\ ax_1 \ \ |x_1| \leqslant 1 \\ bx_1 - a + b \ x_1 \leqslant -1 \end{cases} \tag{11}
$$

where the real parameters α and β determined by the particular values of the circuit components are in a standard model $\alpha = 9$, $\beta = 100/7$, $a = -8/7$ and $b = -5/7$ and where the functions f_i are infinitely differentiable with respect to all x_i, and t, i.e., are C^{∞} functions in a compact E included in $\mathbb{R}^3$ and with values in $\mathbb{R}$.

3.1.1 Tangent linear system approximation

The piecewise linear Chua's circuit has three fixed points around which the *double scrolls* wind. Thus, each *scroll* lies on a *plane* passing through a fixed a point. Its equation may be calculated while using the *Tangent Linear System Approximation* [Rossetto, 1993] which consists in using the *fast* eigenvector associated with the *fast* eigenvalue of the transposed functional Jacobian matrix in order to define the normal vector to these *planes*.

The transposed functional jacobian matrix of Chua's system (10) reads:

$$
{}^tJ = \begin{pmatrix} -\alpha(1+b) & 1 & 0 \\ \alpha & -1 & -\beta \\ 0 & 1 & 0 \end{pmatrix}
$$

The *fast* eigenvector associated with the *fast* eigenvalue λ_1 may be written:

$$
{}^t\overrightarrow{Y_{\lambda_1}}\begin{pmatrix} 1 \\ \lambda_1 + \alpha(b+1) \\ 1 + \alpha\frac{b+1}{\lambda_1} \end{pmatrix}
$$

Let's denote $\overrightarrow{IM}(x - x_I, y - y_I, z - z_I)$ where I is any fixed point I_1 or I_2 and M any point belonging to the phase space.
It may be checked that: $\overrightarrow{V} = J\overrightarrow{IM}$

Thus, according to this method, the (Π) *plane* equation passing through the fixed point I_1 (resp. I_2) may be given by the following orthogonality condition:

$$
\Pi\left(\vec{X}\right) = \overrightarrow{V} \cdot {}^t\overrightarrow{Y_{\lambda_1}} = 0 \tag{12}
$$

But since $\overrightarrow{V} = J\overrightarrow{IM}$, Eq. (12) reads: $\Pi\left(\vec{X}\right) = J\overrightarrow{IM} \cdot {}^t\overrightarrow{Y_{\lambda_1}} = 0$.

Then, according to the *eigenequation*: ${}^tJ\,{}^t\overrightarrow{Y_{\lambda_1}} = \lambda_1 {}^t\overrightarrow{Y_{\lambda_1}}$, it may be checked that:

$$\vec{V} \cdot {}^t\overrightarrow{Y_{\lambda_1}} = \lambda_1 {}^t\overrightarrow{Y_{\lambda_1}} \cdot \overrightarrow{IM} \tag{13}$$

So, the (Π) *plane* equation passing through the fixed point I_1 (resp. I_2) is given by:

$$\Pi\left(\vec{X}\right) = \lambda_1 \overrightarrow{IM} \cdot {}^t\overrightarrow{Y_{\lambda_1}} = 0 \tag{14}$$

The Lie derivative of $\Pi\left(\vec{X}\right)$ reads, taking into account Eq. (13) & Eq. (14):

$$L_{\vec{V}}\Pi\left(\vec{X}\right) = \lambda_1 \frac{d\overrightarrow{IM}}{dt} \cdot {}^t\overrightarrow{Y_{\lambda_1}} = \lambda_1 \vec{V} \cdot {}^t\overrightarrow{Y_{\lambda_1}} = \lambda_1\left(\lambda_1 \overrightarrow{IM} \cdot {}^t\overrightarrow{Y_{\lambda_1}}\right) = \lambda_1 \Pi\left(\vec{X}\right)$$

So, according to *Darboux Theorem* [1878], the *plane* $\Pi\left(\vec{X}\right)$ is invariant.

3.1.2 Curvature of the flow

Curvature of the flow states that the location of the points where the *second curvature* (*torsion*) of the flow, i.e., the *second curvature* of the *trajectory curves* integral of Chua's system vanishes directly provides its *slow invariant manifold* analytical equation, i.e., the (Π) *planes* equations. According to Proposition 3.1, Eq. (2) may be written:

$$\phi\left(\vec{X}\right) = \vec{V} \cdot \left(\vec{\gamma} \wedge \dot{\vec{\gamma}}\right) = 0 \tag{15}$$

It can been easily established for any dynamical system that: $\vec{\gamma} = J\vec{V}$. Moreover, since the Chua's system (10) is piecewise linear the time derivative of the functional jacobian matrix is zero: $\frac{dJ}{dt} = 0$. As a consequence, the over-acceleration (or jerk) reads: $\dot{\vec{\gamma}} = J\vec{\gamma} + \frac{dJ}{dt}\vec{V} = J\vec{\gamma}$.

But since $\vec{V} = J\overrightarrow{IM}$, the *slow manifold* equation (15) may be written:

$$\phi\left(\vec{X}\right) = J\overrightarrow{IM} \cdot \left(J\vec{V} \wedge J\vec{\gamma}\right) = 0 \tag{16}$$

The identity (59) $J\vec{a}.\left(J\vec{b} \wedge J\vec{c}\right) = Det\left(J\right)\vec{a}.\left(\vec{b} \wedge \vec{c}\right)$ established in appendix leads to:

$$\phi\left(\vec{X}\right) = Det\left(J\right)\overrightarrow{IM} \cdot \left(\vec{V} \wedge \vec{\gamma}\right) = 0 \tag{17}$$

where, $\overrightarrow{IM} \cdot \left(\vec{V} \wedge \vec{\gamma}\right) = 0$ is the *osculating plane* passing through the fixed point I_1 (resp. I_2).

The Lie derivative of $\phi\left(\vec{X}\right)$ reads, taking into account that $\dot{\vec{\gamma}} = J\vec{\gamma}$

$$L_{\vec{V}}\phi\left(\vec{X}\right) = Det\left(J\right)\overrightarrow{IM}\cdot\left(\vec{V}\wedge\dot{\vec{\gamma}}\right) = Det\left(J\right)\overrightarrow{IM}\cdot\left(\vec{V}\wedge J\vec{\gamma}\right) = 0 \tag{18}$$

The identity (61) $J\vec{a}.\left(\vec{b}\wedge\vec{c}\right) + \vec{a}.\left(J\vec{b}\wedge\vec{c}\right) + \vec{a}.\left(\vec{b}\wedge J\vec{c}\right) = Tr\left(J\right)\vec{a}.\left(\vec{b}\wedge\vec{c}\right)$ established in appendix leads to:

$$\overrightarrow{IM}\cdot\left(\vec{V}\wedge J\vec{\gamma}\right) = Tr\left(J\right)\overrightarrow{IM}\cdot\left(\vec{V}\wedge\vec{\gamma}\right) \quad \text{and} \quad L_{\vec{V}}\phi\left(\vec{X}\right) = Tr\left[J\right]\phi\left(\vec{X}\right).$$

So, according to *Darboux Theorem* [1878], the manifold $\phi\left(\vec{X}\right)$ is invariant.

Moreover, while multiplying Eq. (12) by its "conjugated" equations, i.e., by $\vec{V}\cdot{}^{t}\overrightarrow{Y_{\lambda_2}}$ and $\vec{V}\cdot{}^{t}\overrightarrow{Y_{\lambda_3}}$ we have:

$$\left(\vec{V}\cdot{}^{t}\overrightarrow{Y_{\lambda_1}}\right)\left(\vec{V}\cdot{}^{t}\overrightarrow{Y_{\lambda_2}}\right)\left(\vec{V}\cdot{}^{t}\overrightarrow{Y_{\lambda_3}}\right) = 0 \tag{19}$$

But, it has been established [Ginoux *et al.*, 2006] that Eq. (19) is totally identical to Eq. (15). So, taking into account Eq. (13) & (14), it may be written:

$$\Pi\left(\vec{X}\right)\left(\vec{V}\cdot{}^{t}\overrightarrow{Y_{\lambda_2}}\right)\left(\vec{V}\cdot{}^{t}\overrightarrow{Y_{\lambda_3}}\right) = \vec{V}\cdot\left(\vec{\gamma}\wedge\dot{\vec{\gamma}}\right) = 0$$

Then, it proves that the (Π) *plane* equation (14) is in factor in Eq. (15) and so that both methods provide the same *planes* equations. Moreover, in the framework of *Differential Geometry*, the (Π) *plane* may be interpreted as the *osculating plane* passing through each fixed point I_1 (resp. I_2).

With this set of parameters: $\lambda_1 = -3.9421$; ${}^{t}\overrightarrow{Y_{\lambda_1}}\left(2.8759, -3.9421, 1\right)$

($\Pi_{1,2}$) *hyperplanes* equations passing through the fixed point $I_{1,2}\left(\mp 3/2, 0, \pm 3/2\right)$ given by both methods read:

$$\Pi_{1,2}\left(\vec{X}\right) = 2.8759x_1 - 3.9421x_2 + x_3 \pm 2.8139 = 0$$

and are plotted in Fig. 1.

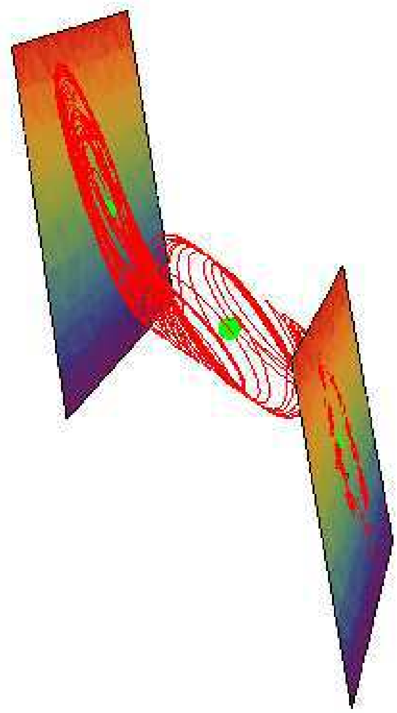

Figure 1: Chua's chaotic invariant hyperplanes for: $\alpha = 9$, $\beta = 100/7$, $a = -8/7$, $b = -5/7$.

3.2 Four-dimensional Chua's system

The piecewise linear fourth-order Chua's circuit [Thamilmaran *et al.*, 2004] is an electronic circuit comprising two inductances L_1 and L_2, two linear resistors R and R_1, two capacitors C_1 and C_2, and a nonlinear resistor. Fourth-order Chua's circuit can be accurately modeled by means of a system of four coupled first-order ordinary differential equations in the variables $x_1(t)$, $x_2(t)$, $x_3(t)$ and $x_4(t)$, which give the voltages in the capacitors C_1 and C_2, and the intensities of the electrical current in the inductance L_1 and L_2, respectively. These equations called *global unfolding* of Chua's circuit are written in a dimensionless form:

$$\vec{V}\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \\ \frac{dx_4}{dt} \end{pmatrix} = \vec{\Im}\begin{pmatrix} f_1(x_1, x_3, x_3, x_4) \\ f_2(x_1, x_3, x_3, x_4) \\ f_3(x_1, x_3, x_3, x_4) \\ f_4(x_1, x_3, x_3, x_4) \end{pmatrix} = \begin{pmatrix} \alpha_1(x_3 - k(x_1)) \\ \alpha_2 x_2 - x_3 - x_4 \\ \beta_1(x_2 - x_1 - x_3) \\ \beta_2 x_2 \end{pmatrix} \tag{20}$$

The function $k(x_1)$ describes the electrical response of the nonlinear resistor, i.e., its characteristics which is a piecewise linear function defined by:

$$k(x_1) = \begin{cases} bx_1 + a - b \;\; x_1 \geqslant 1 \\ ax_1 \;\; |x_1| \leqslant 1 \\ bx_1 - a + b \;\; x_1 \leqslant -1 \end{cases} \tag{21}$$

where the real parameters α_i and β_i determined by the particular values of the circuit components are in a standard model $\alpha_1 = 2.1429$, $\alpha_2 = -0.18$, $\beta_1 = 0.0774$, $\beta_2 = 0.003$ $a = -0.42$, $b = 1.2$ and where the functions f_i are infinitely differentiable with respect to all x_i, and t, i.e., are C^∞ functions in a compact E included in $\mathbb{R}^4$ and with values in $\mathbb{R}$.

3.2.1 Tangent linear system approximation

The fourth-order piecewise linear Chua's circuit has three fixed points around which the *double scroll* winds in a hyperspace of dimension four. In a reduced phase space of dimension three, each *scroll* lies on a *hyperplane* passing through a fixed a point the equation of which may be calculated while using the *Generalized Tangent Linear System Approximation* presented in appendix. So, according to this method, the (Π) *hyperplane* equation passing through the fixed point I_1 (resp. I_2) is given by the following orthogonality condition:

$$\Pi\left(\vec{X}\right) = \vec{V} \cdot {}^t\overrightarrow{Y_{\lambda_1}} = 0 \tag{22}$$

The piecewise linear feature enables to extend the results of the previous Sec. 3.1. to higher dimensions. So, the (Π) *hyperplanes* equations passing through the fixed point I_1 (resp. I_2) is given by:

$$\Pi\left(\vec{X}\right) = \lambda_1 \overrightarrow{IM} \cdot {}^t\overrightarrow{Y_{\lambda_1}} = 0 \tag{23}$$

The Lie derivative of $\Pi\left(\vec{X}\right)$ reads:

$$L_{\vec{V}}\Pi\left(\vec{X}\right) = \lambda_1 \frac{d\overrightarrow{IM}}{dt} \cdot {}^t\overrightarrow{Y_{\lambda_1}} = \lambda_1 \overrightarrow{V} \cdot {}^t\overrightarrow{Y_{\lambda_1}} = \lambda_1 \left(\lambda_1 \overrightarrow{IM} \cdot {}^t\overrightarrow{Y_{\lambda_1}}\right) = \lambda_1 \Pi\left(\vec{X}\right)$$

So, according to *Darboux Theorem* [1878], the *hyperplane* $\Pi\left(\vec{X}\right)$ is invariant.

3.2.2 Curvature of the flow

Curvature of the flow states that the location of the points where the *third curvature* of the flow, i.e., the *third curvature* of the *trajectory curves* integral of Chua's fourth-order system vanishes directly provides its *slow invariant manifold* analytical equation, i.e., the (Π) *hyperplanes* equations. According to Proposition 3.1, Eq. (2) may be written:

$$\phi\left(\vec{X}\right) = \overrightarrow{V} \cdot \left(\vec{\gamma} \wedge \dot{\vec{\gamma}} \wedge \ddot{\vec{\gamma}}\right) = 0 \tag{24}$$

The piecewise linear feature enables to state that: $\overset{(n)}{\vec{\gamma}} = J^{(n+1)}\overrightarrow{V} = J^{(n)}\vec{\gamma}$. So, according to the fact that as previously: $\overrightarrow{V} = J\overrightarrow{IM}$, the *slow manifold* equation (24) reads:

$$\phi\left(\vec{X}\right) = J\overrightarrow{IM} \cdot \left(J\overrightarrow{V} \wedge J\vec{\gamma} \wedge J\dot{\vec{\gamma}}\right) = 0 \tag{25}$$

Identity (59) $J\vec{a}.\left(J\vec{b} \wedge J\vec{c} \wedge J\vec{d}\right) = Det\left(J\right)\vec{a}.\left(\vec{b} \wedge \vec{c} \wedge \vec{d}\right)$ established in appendix leads to:

$$\phi\left(\vec{X}\right) = Det\left(J\right)\overrightarrow{IM} \cdot \left(\overrightarrow{V} \wedge \vec{\gamma} \wedge \dot{\vec{\gamma}}\right) = 0 \tag{26}$$

where, $\overrightarrow{IM} \cdot \left(\overrightarrow{V} \wedge \vec{\gamma} \wedge \dot{\vec{\gamma}}\right) = 0$ is the *osculating plane* passing through the fixed point I_1 (resp. I_2). The Lie derivative of $\phi\left(\vec{X}\right)$ reads, taking into account that $\dot{\vec{\gamma}} = J\vec{\gamma}$ and $\ddot{\vec{\gamma}} = J\dot{\vec{\gamma}}$

$$L_{\vec{V}}\phi\left(\vec{X}\right) = Det\left(J\right)\overrightarrow{IM} \cdot \left(\overrightarrow{V} \wedge \vec{\gamma} \wedge \ddot{\vec{\gamma}}\right) = Det\left(J\right)\overrightarrow{IM} \cdot \left(\overrightarrow{V} \wedge \vec{\gamma} \wedge J\dot{\vec{\gamma}}\right) = 0 \tag{27}$$

The identity (61) established in appendix:

$$J\vec{a}.\left(\vec{b} \wedge \vec{c} \wedge \vec{d}\right) + \vec{a}.\left(J\vec{b} \wedge \vec{c} \wedge \vec{d}\right) + \vec{a}.\left(\vec{b} \wedge J\vec{c} \wedge \vec{d}\right) + \vec{a}.\left(\vec{b} \wedge \vec{c} \wedge J\vec{d}\right) = Tr\left(J\right)\vec{a}.\left(\vec{b} \wedge \vec{c} \wedge \vec{d}\right)$$

leads to: $\overrightarrow{IM} \cdot \left(\overrightarrow{V} \wedge \vec{\gamma} \wedge J\dot{\vec{\gamma}}\right) = Tr\left(J\right)\overrightarrow{IM} \cdot \left(\overrightarrow{V} \wedge \vec{\gamma} \wedge \dot{\vec{\gamma}}\right)$ and $L_{\vec{V}}\phi\left(\vec{X}\right) = Tr\left[J\right]\phi\left(\vec{X}\right)$.

Moreover, while multiplying $\overrightarrow{V} \cdot {}^t\overrightarrow{Y_{\lambda_1}}$ by its "conjugated" equations, i.e., by $\overrightarrow{V} \cdot {}^t\overrightarrow{Y_{\lambda_2}}$, $\overrightarrow{V} \cdot {}^t\overrightarrow{Y_{\lambda_3}}$ and $\overrightarrow{V} \cdot {}^t\overrightarrow{Y_{\lambda_4}}$ we have:

$$\left(\overrightarrow{V} \cdot {}^t\overrightarrow{Y_{\lambda_1}}\right)\left(\overrightarrow{V} \cdot {}^t\overrightarrow{Y_{\lambda_2}}\right)\left(\overrightarrow{V} \cdot {}^t\overrightarrow{Y_{\lambda_3}}\right)\left(\overrightarrow{V} \cdot {}^t\overrightarrow{Y_{\lambda_4}}\right) = 0 \tag{28}$$

It may also be established that Eq. (28) is totally identical to Eq. (24) and so that

$$\Pi\left(\vec{X}\right)\left(\overrightarrow{V} \cdot {}^t\overrightarrow{Y_{\lambda_2}}\right)\left(\overrightarrow{V} \cdot {}^t\overrightarrow{Y_{\lambda_3}}\right)\left(\overrightarrow{V} \cdot {}^t\overrightarrow{Y_{\lambda_4}}\right) = \overrightarrow{V} \cdot \left(\vec{\gamma} \wedge \dot{\vec{\gamma}} \wedge \ddot{\vec{\gamma}}\right) = 0$$

Then, it proves that the (Π) *hyperplane* equation (23) is in factor in Eq. (24) and so that both methods provide the same *hyperplanes* equations. Moreover, in the framework of *Differential Geometry*, the (Π) *hyperplane* may be interpreted as the *osculating hyperplane* passing through each fixed point I_1 (resp. I_2).

With this set of parameters:

- $\lambda_1 = -2.5039$;
- ${}^t\overrightarrow{Y_{\lambda_1}}\,(-0.7532, -0.01895, 0.6574, -0.007568)$

$(\Pi_{1,2})$ *hyperplanes* equations passing through fixed point $I_{1,2}\,(\mp 0.7363, 0, \pm 0.7363, \mp 0.7363)$ given by both methods read:

$$\Pi_{1,2}\left(\vec{X}\right) = 1.8861x_1 + 0.04744x_2 - 1.6461x_3 + 0.01895x_4 \pm 2.6149 = 0$$

and are plotted in Fig. 2.

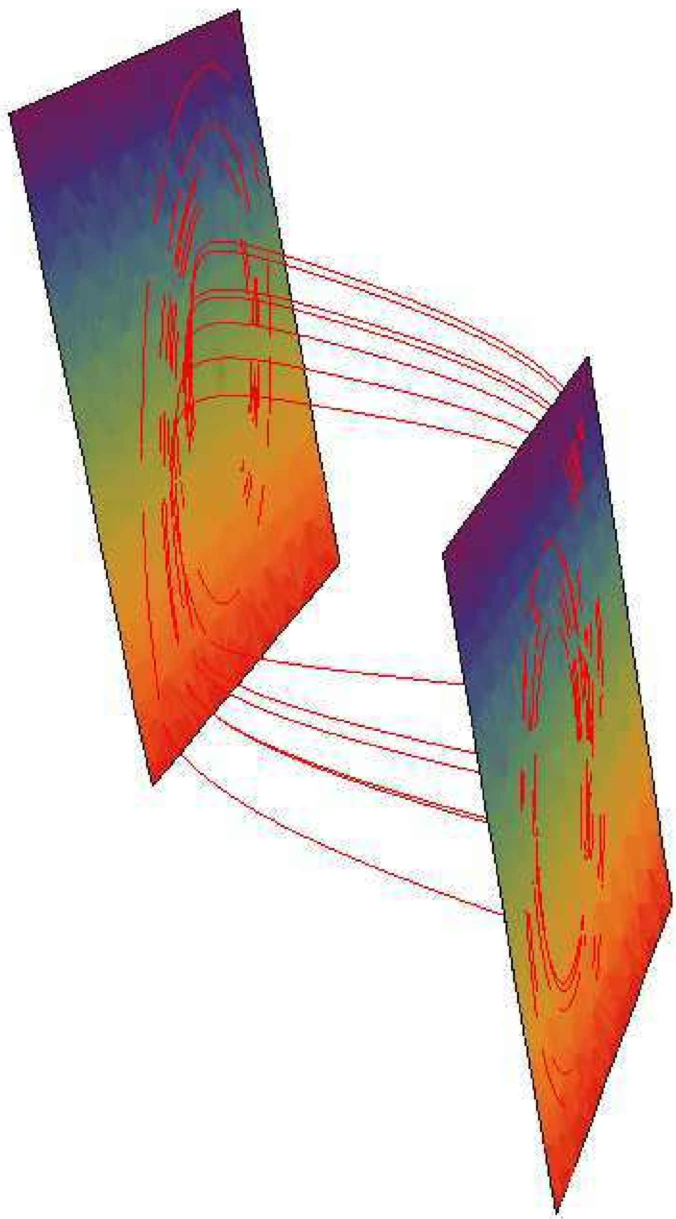

Figure 2: Chua's fourth-order *invariant hyperplanes* in $(x_1 x_2 x_3)$ space for: $\alpha_1 = 2.1429, \alpha_2 = -0.18, \beta_1 = 0.0774, \beta_2 = 0.003, a = -0.42, b = 1.2$.

3.3 Five-dimensional Chua's system

The piecewise linear fifth-order Chua's circuit [Hao *et al.*, 2005] is built while adding a RLC parallel circuit into the L-arm of Chua's circuit. This electronic circuit consists of two inductances L_1 and L_2, two linear resistors R and R_1, three capacitors C_1, C_2 and C_3, and a nonlinear resistor. Fifth-order Chua's circuit can be accurately modeled by means of a system of five coupled first-order ordinary differential equations in the variables $x_1(t)$, $x_2(t)$, $x_3(t)$, $x_4(t)$ and $x_5(t)$, which give the voltages in the capacitors C_1, C_2 and C_3, and the intensities of the electrical current in the inductance L_1 and L_2, respectively. These equations called *global unfolding* of Chua's circuit are written in a dimensionless form:

$$\vec{V}\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \\ \frac{dx_4}{dt} \\ \frac{dx_5}{dt} \end{pmatrix} = \vec{\Im}\begin{pmatrix} f_1(x_1,x_3,x_3,x_4,x_5) \\ f_2(x_1,x_3,x_3,x_4,x_5) \\ f_3(x_1,x_3,x_3,x_4,x_5) \\ f_4(x_1,x_3,x_3,x_4,x_5) \\ f_5(x_1,x_3,x_3,x_4,x_5) \end{pmatrix} = \begin{pmatrix} \alpha_1(x_2 - x_1 - k(x_1)) \\ \alpha_2 x_1 - x_2 + x_3 \\ \beta_1(x_4 - x_2) \\ \beta_2(x_3 + x_5) \\ \gamma_2(x_4 + \gamma_1 x_5) \end{pmatrix} \tag{29}$$

The function $k(x_1)$ describes the electrical response of the nonlinear resistor, i.e., its characteristics which is a piecewise linear function defined by:

$$k(x_1) = \begin{cases} bx_1 + a - b \; x_1 \geqslant 1 \\ ax_1 \;\; |x_1| \leqslant 1 \\ bx_1 - a + b \; x_1 \leqslant -1 \end{cases} \tag{30}$$

where the real parameters α_i, β_i and γ_i determined by the particular values of the circuit components are: $\alpha_1 = 9.934$, $\alpha_2 = 1$, $\beta_1 = 14.47$, $\beta_2 = -406.5$, $\gamma_1 = -0.0152$, $\gamma_2 = 41000$, $a = -1.246$, $b = -0.6724$ and where the functions f_i are infinitely differentiable with respect to all x_i, and t, i.e., are C^∞ functions in a compact E included in $\mathbb{R}^5$ and with values in $\mathbb{R}$.

3.3.1 Tangent linear system approximation

The fifth-order piecewise linear Chua's circuit has three fixed points around which the *double scroll* winds in a hyper space of dimension five. In a reduced phase space of dimension three, each *scroll* lies on a (Π) *hyperplane* equation passing through the fixed point I_1 (resp. I_2) the equation of which may be calculated still using the *Generalized Tangent Linear System Approximation* presented in appendix. So, the following orthogonality condition leads to:

$$\Pi\left(\vec{X}\right) = \vec{V} \cdot {}^t\overrightarrow{Y_{\lambda_1}} = 0 \tag{31}$$

The piecewise linear feature still enables to extend the results of the previous Sec. 3.1. to higher dimensions. So, the (Π) *hyperplanes* passing through the fixed point I_1 (resp. I_2) are invariant according to *Darboux Theorem* [1878].

3.3.2 Curvature of the flow

Curvature of the flow states that the location of the points where the *fourth curvature* of the flow, i.e., the *fourth curvature* of the *trajectory curves* integral of Chua's fifth-order system vanishes directly provides its *slow invariant manifold* analytical equation, i.e., the (Π) *hyperplanes* equations. According to Proposition 3.1, Eq. (2) may be written:

$$\phi\left(\vec{X}\right) = \overrightarrow{V} \cdot \left(\vec{\gamma} \wedge \dot{\vec{\gamma}} \wedge \ddot{\vec{\gamma}} \wedge \dddot{\vec{\gamma}}\right) = 0 \tag{32}$$

The piecewise linear feature and both identity (59) and (61) enable to state, according to *Darboux Theorem* [1878], that the manifold $\phi\left(\vec{X}\right)$ is invariant.

Morover, while multiplying $\overrightarrow{V} \cdot {}^t\overrightarrow{Y_{\lambda_1}}$ by its "conjugated" equations, i.e., by $\overrightarrow{V} \cdot {}^t\overrightarrow{Y_{\lambda_2}}$, $\overrightarrow{V} \cdot {}^t\overrightarrow{Y_{\lambda_3}}$, $\overrightarrow{V} \cdot {}^t\overrightarrow{Y_{\lambda_4}}$ and $\overrightarrow{V} \cdot {}^t\overrightarrow{Y_{\lambda_5}}$ we have:

$$\left(\overrightarrow{V} \cdot {}^t\overrightarrow{Y_{\lambda_1}}\right)\left(\overrightarrow{V} \cdot {}^t\overrightarrow{Y_{\lambda_2}}\right)\left(\overrightarrow{V} \cdot {}^t\overrightarrow{Y_{\lambda_3}}\right)\left(\overrightarrow{V} \cdot {}^t\overrightarrow{Y_{\lambda_4}}\right)\left(\overrightarrow{V} \cdot {}^t\overrightarrow{Y_{\lambda_5}}\right) = 0 \tag{33}$$

It may also be established that Eq. (33) is totally identical to Eq. (32) and so that

$$\Pi\left(\vec{X}\right)\left(\overrightarrow{V} \cdot {}^t\overrightarrow{Y_{\lambda_2}}\right)\left(\overrightarrow{V} \cdot {}^t\overrightarrow{Y_{\lambda_3}}\right)\left(\overrightarrow{V} \cdot {}^t\overrightarrow{Y_{\lambda_4}}\right)\left(\overrightarrow{V} \cdot {}^t\overrightarrow{Y_{\lambda_5}}\right) = \overrightarrow{V} \cdot \left(\vec{\gamma} \wedge \dot{\vec{\gamma}} \wedge \ddot{\vec{\gamma}} \wedge \dddot{\vec{\gamma}}\right) = 0$$

Then, it proves that the (Π) *hyperplane* equation (31) is in factor in Eq. (32) and so that both methods provide the same *hyperplanes* equations. Moreover, in the framework of *Differential Geometry*, the (Π) *hyperplane* may still be interpreted as the *osculating hyperplane* passing through each fixed point I_1 (resp. I_2).

With this set of parameters eigenvalues and eigenvectors are respectively:

- $\lambda_1 = -311.49$
- ${}^t\overrightarrow{Y_{\lambda_1}}\,(0.5625, -0.8068, 0.1804, 0.00009693, -0.000063709)$

$(\Pi_{1,2})$ *hyperplanes* equations passing through fixed point

$$I_{1,2}\,(\mp 1.83477, \mp 0.027471, \pm 1.8073, \mp 0.027471, \mp 1.8073)$$

given by both methods read:

$$\Pi_{1,2}\left(\vec{X}\right) = -2.63746x_1 + 3.78315x_2 - 0.846258x_3 - 0.000454517x_4 + 0.000298719x_5 \mp 3.20524$$

and are plotted in Fig. 3.

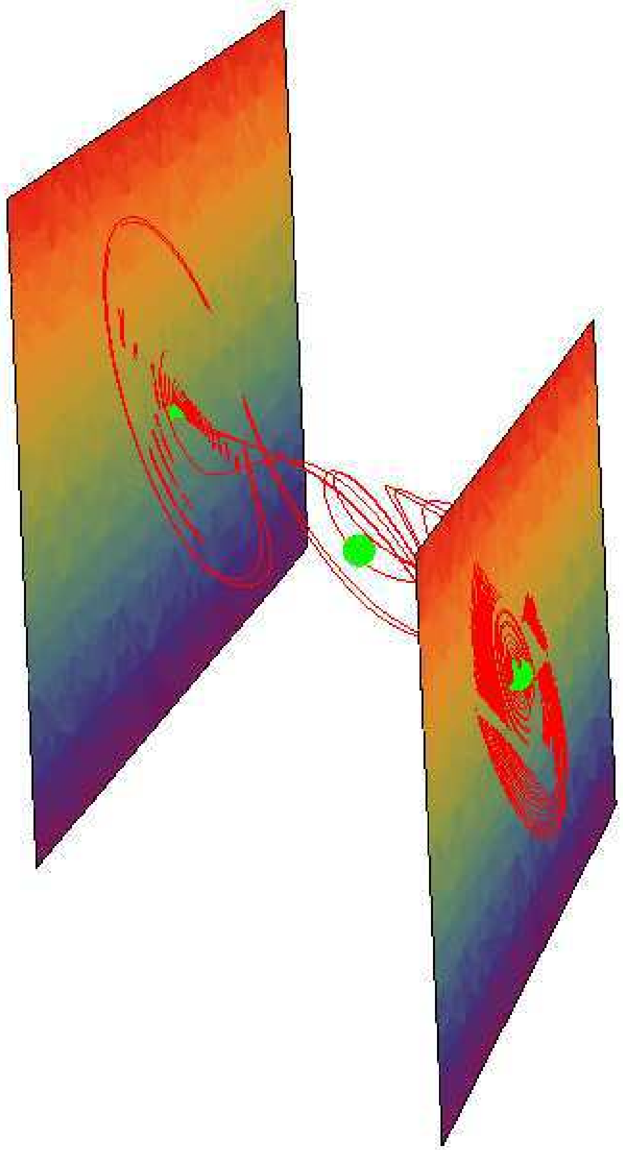

Figure 3: Chua's fifth-order *invariant hyperplanes* in $(x_1x_2x_3)$ space for: $\alpha_2 = 1$, $\alpha_1 = 9.934$, $\beta_1 = 14.47$, $\beta_2 = -406.5$, $\gamma_1 = -0.0152$, $\gamma_2 = 41000$, $a = -1.246$, $b = -0.6724$.

4 Chua's cubic nonlinear models

After these tutorial examples concerning Chua's piecewise linear systems, let's apply the *curvature of the flow* to *nonlinear* Chua's cubic systems of dimension three, four and five.

4.1 Three-dimensional cubic Chua's system

The *slow invariant manifold* of the third-order Chua's cubic circuit [Rossetto *et al.*, 1998] has already been calculated with *curvature of the flow* in [Ginoux *et al.*, 2006].

4.2 Four-dimensional cubic Chua's system

The fourth-order cubic Chua's circuit [Thamilmaran *et al.*, 2004, Liu *et al.*, 2007] may be described starting from the same set of differential equations as (20) but while replacing the piecewise linear function by a smooth cubic nonlinear.

$$\overrightarrow{V}\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \\ \frac{dx_4}{dt} \end{pmatrix} = \overrightarrow{\Im}\begin{pmatrix} f_1\left(x_1, x_3, x_3, x_4\right) \\ f_2\left(x_1, x_3, x_3, x_4\right) \\ f_3\left(x_1, x_3, x_3, x_4\right) \\ f_4\left(x_1, x_3, x_3, x_4\right) \end{pmatrix} = \begin{pmatrix} \alpha_1\left(x_3 - \hat{k}\left(x_1\right)\right) \\ \alpha_2 x_2 - x_3 - x_4 \\ \beta_1\left(x_2 - x_1 - x_3\right) \\ \beta_2 x_2 \end{pmatrix} \tag{34}$$

The function $\hat{k}\left(x_1\right)$ describing the electrical response of the nonlinear resistor is an odd-symmetric function similar to the piecewise linear nonlinearity $k\left(x_1\right)$ for which the parameters $c_1 = 0.3937$ and $c_2 = -0.7235$ are determined while using least-square method [Tsuneda, 2005] and which characteristics is defined by:

$$\hat{k}\left(x_1\right) = c_1 x_1^3 + c_2 x_1 \tag{35}$$

The real parameters α_i and β_i determined by the particular values of the circuit components are in a standard model $\alpha_1 = 2.1429$, $\alpha_2 = -0.18$, $\beta_1 = 0.0774$, $\beta_2 = 0.003$ $c_1 = 0.3937$ and $c_2 = -0.7235$ and where the functions f_i are infinitely differentiable with respect to all x_i, and t, i.e., are C^∞ functions in a compact E included in $\mathbb{R}^4$ and with values in $\mathbb{R}$.

Curvature of the flow states that the location of the points where the *fourth curvature of the flow*, i.e., the *fourth curvature* of the *trajectory curves* integral of Chua's cubic system vanishes directly provides its *slow invariant manifold* analytical equation. According to Proposition 3.1, Eq. (2) may be written:

$$\phi\left(\vec{X}\right) = \overrightarrow{V} \cdot \left(\vec{\gamma} \wedge \dot{\vec{\gamma}} \wedge \ddot{\vec{\gamma}} \wedge \dddot{\vec{\gamma}}\right) = 0 \tag{36}$$

Then, it may be proved that in the vicinity of the *singular approximation* defined by $f_1\left(\vec{X}\right) = 0$ the functional jacobian matrix is stationary, i.e., its time derivative vanishes identically and so, Lie derivative $L_{\vec{V}}\phi\left(\vec{X}\right) = 0$ vanishes identically. Thus, according to *Darboux Theorem* [1878], the manifold $\phi\left(\vec{X}\right)$ which is *locally invariant* is plotted in Fig. 4.

4.3 Five-dimensional models

The fifth-order cubic Chua's circuit [Hao *et al.*, 2005] may be described starting from the same set of differential equations as (29) but while replacing the piecewise linear function by a smooth cubic nonlinear.

$$\overrightarrow{V}\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \\ \frac{dx_4}{dt} \\ \frac{dx_5}{dt} \end{pmatrix} = \overrightarrow{\Im}\begin{pmatrix} f_1\left(x_1, x_3, x_3, x_4, x_5\right) \\ f_2\left(x_1, x_3, x_3, x_4, x_5\right) \\ f_3\left(x_1, x_3, x_3, x_4, x_5\right) \\ f_4\left(x_1, x_3, x_3, x_4, x_5\right) \\ f_5\left(x_1, x_3, x_3, x_4, x_5\right) \end{pmatrix} = \begin{pmatrix} \alpha_1\left(x_2 - x_1 - \hat{k}\left(x_1\right)\right) \\ \alpha_2 x_1 - x_2 + x_3 \\ \beta_1\left(x_4 - x_2\right) \\ \beta_2\left(x_3 + x_5\right) \\ \gamma_2\left(x_4 + \gamma_1 x_5\right) \end{pmatrix} \tag{37}$$

The function $\hat{k}\left(x_1\right)$ describing the electrical response of the nonlinear resistor is an odd-symmetric function similar to the piecewise linear nonlinearity $k\left(x_1\right)$ for which the parameters $c_1 = 0.1068$ and $c_2 = -0.3056$ are determined while using least-square method [Tsuneda, 2005] and which characteristics is defined by:

$$\hat{k}\left(x_1\right) = c_1 x_1^3 + c_2 x_1 \tag{38}$$

The real parameters α_i, β_i and γ_i determined by the particular values of the circuit components are: $\alpha_1 = 9.934$, $\alpha_2 = 1$, $\beta_1 = 14.47$, $\beta_2 = -406.5$,
$\gamma_1 = -0.0152$, $\gamma_2 = 41000$, $a = -1.246$, $b = -0.6724$, $c_1 = 0.1068$, $c_2 = -0.3056$ and where the functions f_i are infinitely differentiable with respect to all x_i, and t, i.e., are C^∞ functions in a compact E included in $\mathbb{R}^5$ and with values in $\mathbb{R}$.

Curvature of the flow states that the location of the points where the *fourth curvature of the flow*, i.e., the *fourth curvature* of the *trajectory curves* integral of Chua's fifth-order system vanishes directly provides its *slow invariant manifold* analytical equation. According to Proposition 3.1, Eq. (2) may be written:

$$\phi\left(\vec{X}\right) = \overrightarrow{V}\cdot\left(\vec{\gamma}\wedge\dot{\vec{\gamma}}\wedge\ddot{\vec{\gamma}}\wedge\dddot{\vec{\gamma}}\wedge\ddddot{\vec{\gamma}}\right) = 0 \tag{39}$$

Then, it may be proved that in the vicinity of the *singular approximation* defined by $f_1\left(\vec{X}\right) = 0$ the functional jacobian matrix is stationary, i.e., its time derivative vanishes identically and so, Lie derivative $L_{\vec{V}}\phi\left(\vec{X}\right) = 0$ vanishes identically. Thus, according to *Darboux Theorem* [1878], the manifold $\phi\left(\vec{X}\right)$ which is *locally invariant* is plotted in Fig. 5.

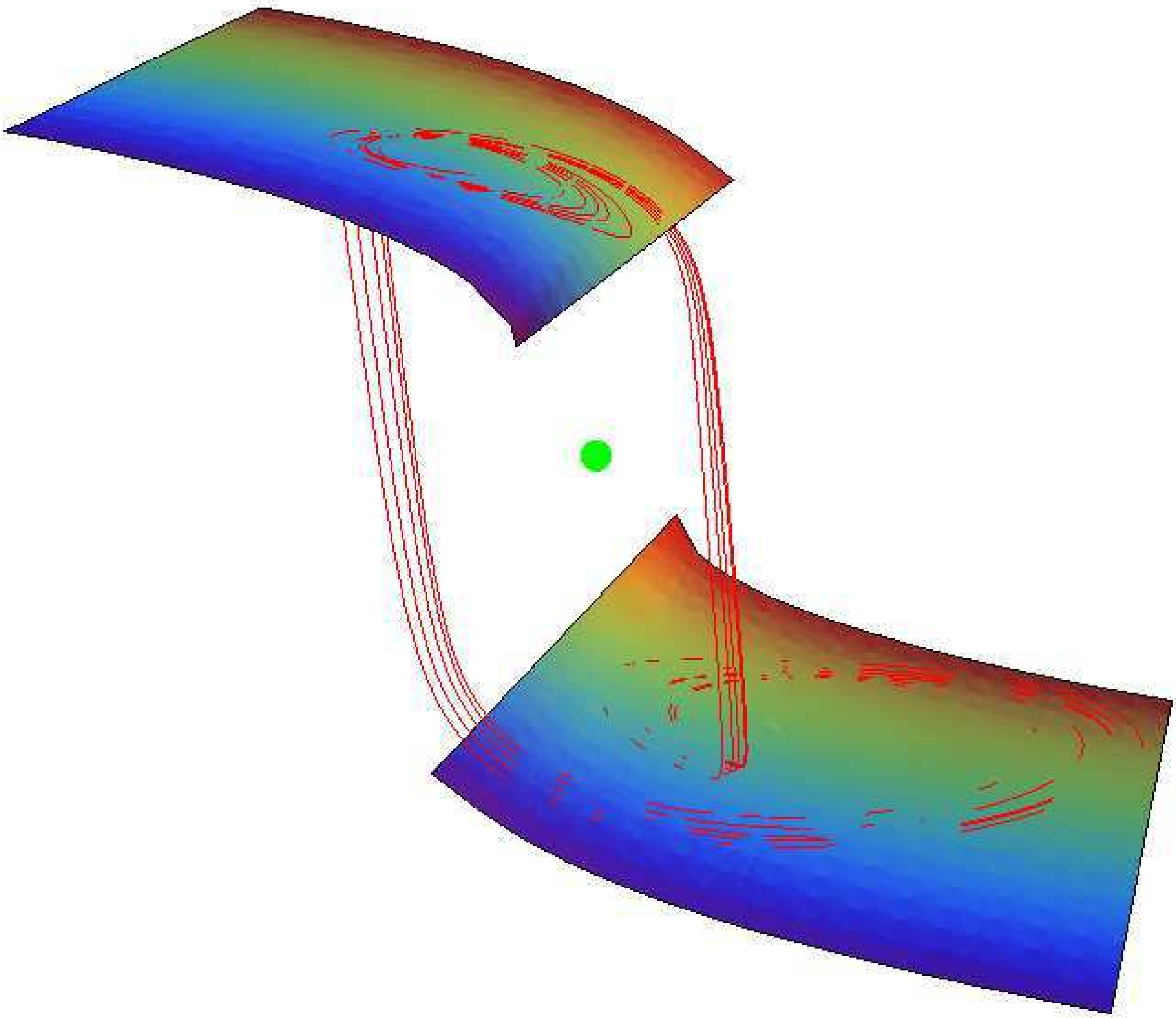

Figure 4: Fourth-order Chua's cubic *invariant manifold* in the $(x_1 x_2 x_3)$ space for: $\alpha_1 = 2.1429$, $\alpha_2 = -0.18$, $\beta_1 = 0.0774$, $\beta_2 = 0.003$, $a = -0.42$, $b = 1.2$.

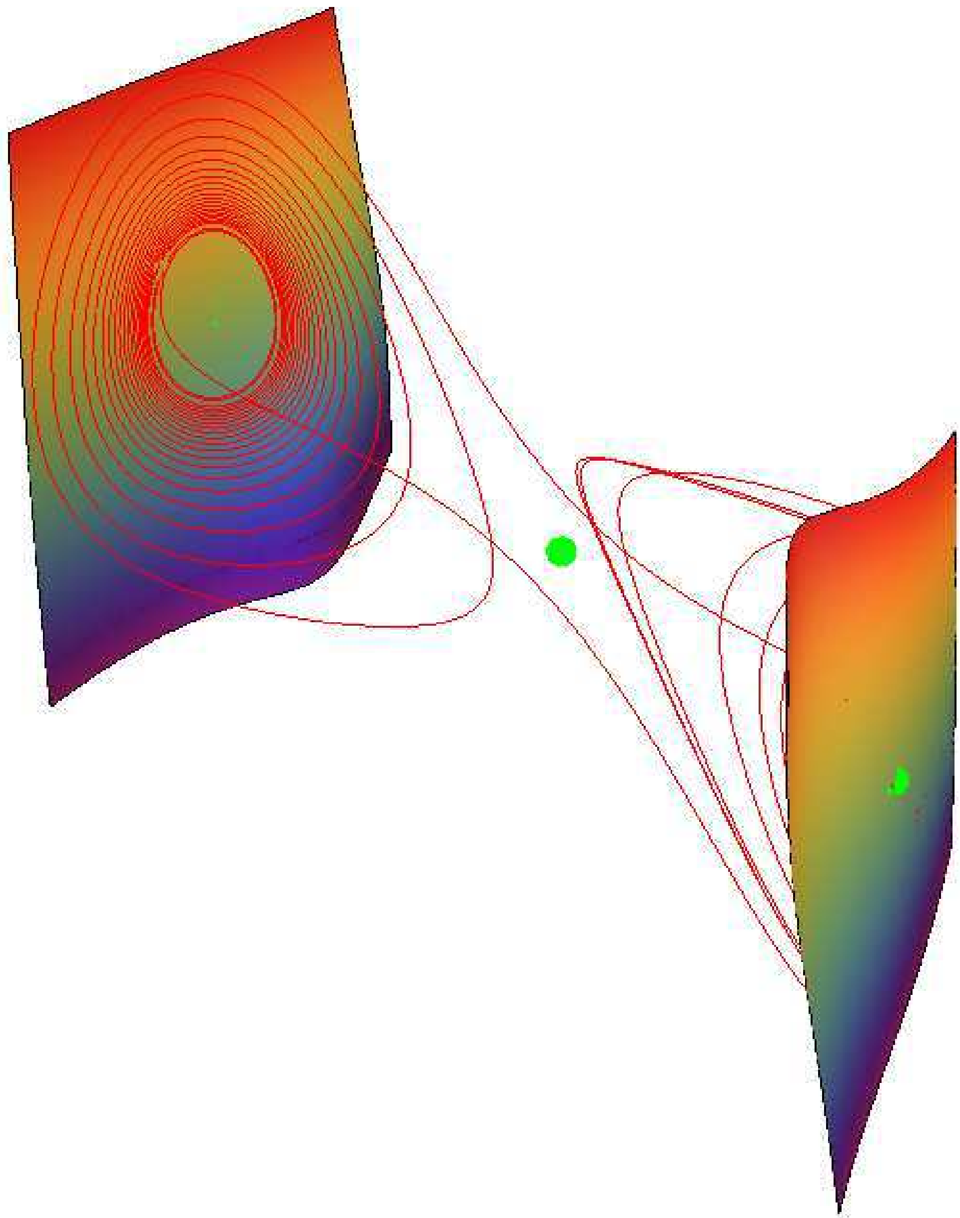

Figure 5: Fifth-order Chua's cubic *invariant manifold* in the $(x_1 x_2 x_3)$ space for: $\alpha_2 = 1$, $\alpha_1 = 9.934$, $\beta_1 = 14.47$, $\beta_2 = -406.5$, $\gamma_1 = -0.0152$, $\gamma_2 = 41000$, $c_1 = 0.1068$, $c_2 = -0.3056$.

5 High-dimensional nonlinear models

In this section two examples are considered. The former is a nonlinear fifth-order model of magnetoconvection [Knobloch *et al.*, 1981] for which the slow invariant manifold will be directly provided by using *curvature of the flow*. The latter is an artificial nonlinear fifth-order model [Gear *et al.*, 2005] having three *attractive invariant manifolds*.

5.1 Five-dimensional magnetoconvection model

A fifth-order system for magnetoconvection [Knobloch *et al.*, 1981] is designed to describe nonlinear coupling between Rayleigh-Bernard convection and an external magnetic field. This type of system was first presented by Veronis [Veronis, 1966] in studying a rotating fluid. The fifth-order system of magnetoconvection is a straightforward extension of the Lorenz model for the Boussinesq convection interacting with the magnetic field. The fifth-order autonomous system of magnetoconvection is given as follows:

$$\vec{V}\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \\ \frac{dx_4}{dt} \\ \frac{dx_5}{dt} \end{pmatrix} = \begin{pmatrix} \sigma\left[-x_1 + rx_2 - qx_4\left(1 + \frac{\omega(3-\omega)}{\varsigma^2(4-\omega)}x_5\right)\right] \\ -x_2 + x_1 - x_1x_3 \\ \omega\left(-x_3 + x_1x_2\right) \\ -\varsigma\left(x_4 - x_1\right) - \frac{\omega}{\varsigma(4-\omega)}x_1x_5 \\ -\varsigma\left(4-\omega\right)\left(x_5 - x_1x_4\right) \end{pmatrix} \tag{40}$$

where $x_1(t)$ represents the first-order velocity perturbation, while$x_2(t)$, $x_3(t)$, $x_4(t)$ and $x_5(t)$ are measures of the first- and the second-order perturbations to the temperature and to the magnetic flux function, respectively. With the five real parameters where $\varsigma = 0.09683$ is the magnetic *Prandtl* number (the ratio of the magnetic to the thermal diffusivity), $\sigma = 1$ is the *Prandtl* number, $r = 14.47$ is a normalized *Rayleigh* number, $q = 5$ is a normalized *Chandrasekhar* number, and $\omega = 0.1081$ is a geometrical parameter and where the functions f_i are infinitely differentiable with respect to all x_i, and t, i.e., are C^∞ functions in a compact E included in $\mathbb{R}^5$ and with values in $\mathbb{R}$.

Curvature of the flow states that the location of the points where the *fourth curvature of the flow*, i.e., the *fourth curvature* of the *trajectory curves* integral of fifth-order magnetoconvection system vanishes directly provides its *slow invariant manifold* analytical equation. According to Proposition 3.1, Eq. (2) may be written:

$$\phi\left(\vec{X}\right) = \vec{V}\cdot\left(\vec{\gamma}\wedge\dot{\vec{\gamma}}\wedge\ddot{\vec{\gamma}}\wedge\dddot{\vec{\gamma}}\wedge\ddddot{\vec{\gamma}}\right) = 0 \tag{41}$$

Then, it may be proved that in the vicinity of the *singular approximation* defined by $f_1\left(\vec{X}\right) = 0$ the functional jacobian matrix is stationary, i.e., its time derivative vanishes identically and so, Lie derivative $L_{\vec{V}}\phi\left(\vec{X}\right) = 0$ vanishes identically. Thus, according to *Darboux Theorem* [1878], the manifold $\phi\left(\vec{X}\right)$ which is *locally invariant* is plotted in Fig. 6.

5.2 Five-dimensional nonlinear model

Let's consider the following fifth-order nonlinear dynamical system [Gear *et al.*, 2005]

$$\overrightarrow{V} = \begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \\ \frac{dx_4}{dt} \\ \frac{dx_5}{dt} \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \\ L\left(x_1^2 + x_2^2 - x_3\right) \\ \beta_1 + x_4^2 \\ \beta_2 + x_2^2 \end{pmatrix} \tag{42}$$

where the real parameters values may be arbitrarily chosen as $L = 1000$, $\beta_1 = 800$, $\beta_2 = 1200$ and where the functions f_i are infinitely differentiable with respect to all x_i, and t, i.e., are C^∞ functions in a compact E included in $\mathbb{R}^5$ and with values in $\mathbb{R}$.

Curvature of the flow states that the location of the points where the *fourth curvature of the flow*, i.e., the *fourth curvature* of the *trajectory curves* integral of fifth-order nonlinear dynamical system vanishes directly provides its *invariant manifolds* analytical equation. According to Proposition 3.1, Eq. (2) may be written:

$$\begin{aligned} \phi\left(\vec{X}\right) &= \overrightarrow{V} \cdot \left(\vec{\gamma} \wedge \dot{\vec{\gamma}} \wedge \ddot{\vec{\gamma}} \wedge \dddot{\vec{\gamma}} \wedge \ddddot{\vec{\gamma}}\right) \\ &= \left(x_1^2 + x_2^2\right)\left(x_1^2 + x_2^2 - x_3\right)\left(x_4^2 + \beta_1\right) Q\left(\vec{X}\right) = 0 \end{aligned} \tag{43}$$

where $Q\left(\vec{X}\right)$ is an irreducible polynomial and while posing:

$$\varphi\left(\vec{X}\right) = \left(x_1^2 + x_2^2\right)\left(x_1^2 + x_2^2 - x_3\right)\left(x_4^2 + \beta_1\right)$$

it may be established that:

$$\begin{aligned} L_{\vec{V}}\varphi\left(\vec{X}\right) &= -\left(L - 2x_4\right)\left(x_1^2 + x_2^2\right)\left(x_1^2 + x_2^2 - x_3\right)\left(x_4^2 + \beta_1\right) \\ &= K\left(\vec{X}\right)\varphi\left(\vec{X}\right) \end{aligned} \tag{44}$$

Thus, according to *Darboux Theorem* [1878], the five-dimensional model has three invariant manifolds, namely $\varphi\left(\vec{X}\right)$ is *invariant*. Moreover, it may be proved that $\left(x_1^2 + x_2^2\right)$ is first integral. So, *curvature of the flow* may also be used to "detect" first integral of dynamical systems.

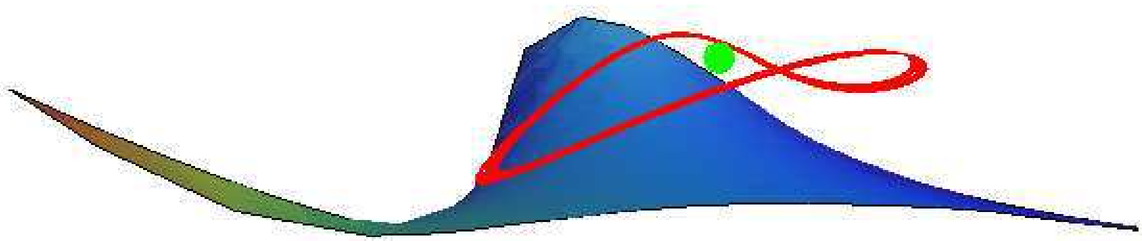

Figure 6: Fifth-order magnetoconvection *invariant manifold* in the $(x_1x_2x_3)$ space for: $\varsigma = 0.09683$, $\sigma = 1$, $r = 14.47$, $q = 5$, $\omega = 0.1081$.

6 Slow invariant manifolds gallery

In this section two examples of *slow invariant manifolds* of chaotic attractors are presented. The first (Fig. 7a.) is a chemical kinetics model used by Gaspard and Nicolis (Journal of Statistics Physics, Vol. 32, Nř 3, 1983, 499 - 518). The second (Fig. 7b) is a neuronal bursting model elaborated by Hindmarsh-Rose (Philos. Trans. Roy. Soc. London Ser. B 221, 1984, 87-102).

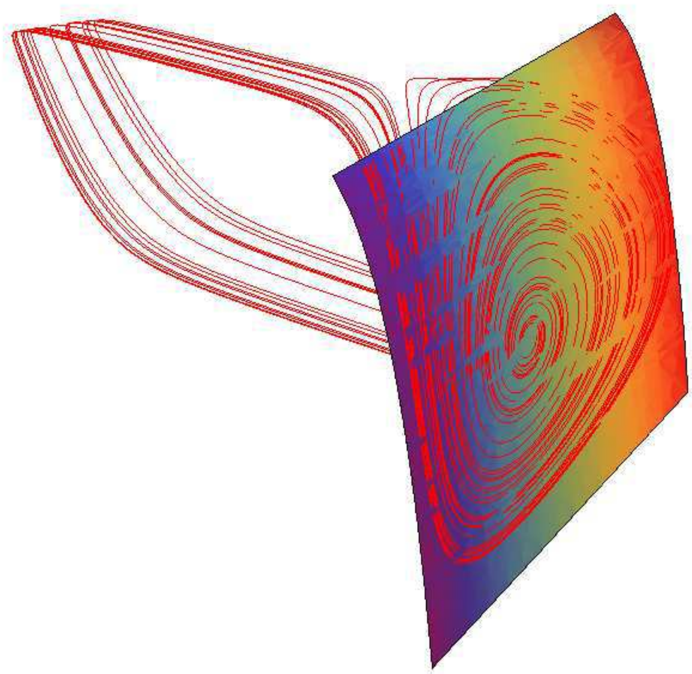

Figure 7: Chemical kinetics model

A gallery of *slow invariant manifolds* is accessible at: http://ginoux.univ-tln.fr

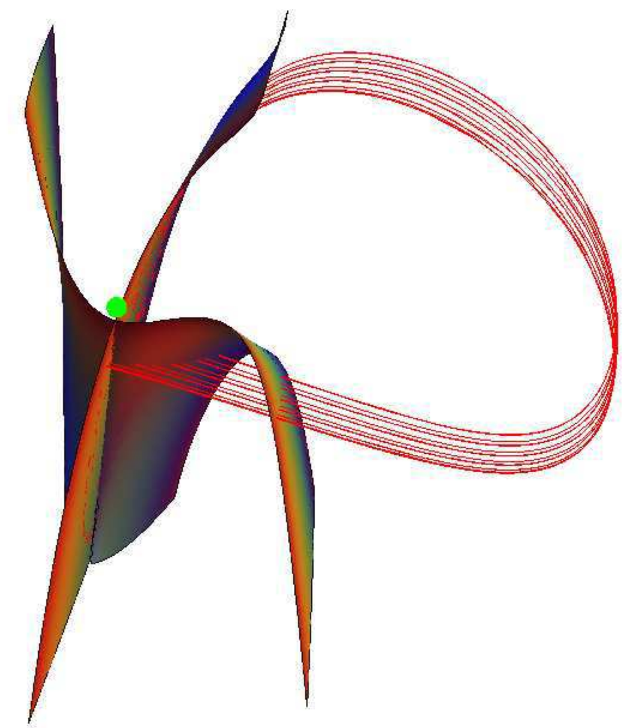

Figure 8: Neuronal bursting model

7 Discussion

During the twentieth century various methods have been developed in order to determine the *slow invariant manifold* analytical equation associated to *slow-fast dynamical systems* or *singularly perturbed systems* among which the so-called *Geometric Singular Perturbation Theory* [Fenichel 1979] and the *Tangent Linear System Approximation* [Rossetto *et al.* 1998]. As pointed out by O'Malley [1974 p. 78, 1991 p. 21] the problem for finding the *slow invariant manifold* analytical equation with the *Geometric Singular Perturbation Theory* turned into a regular perturbation problem in which one generally expected the asymptotic validity of such expansion to breakdown. Moreover, for high-dimensional *singularly perturbed systems slow invariant manifold* analytical equation determination lead to tedious calculations. The *Tangent Linear System Approximation* the generalization of which is presented in appendix, provided the *slow manifold* analytical equation of n-dimensional dynamical systems according to the "slow" eigenvectors of the *tangent linear system*, i.e., according to the "slow" eigenvalues. Nevertheless, the presence of these eigenvalues (real or complex conjugated) prevented from expressing this equation explicitly. Moreover, starting from dimension five *Galois Theory* precludes from analytically computing eigenvalues associated with the functional jacobian matrix of a five-dimensional dynamical system.

In this work, while considering *trajectory curves*, integral of n-dimensional dynamical systems, within the framework of *Differential Geometry* as curves in Euclidean n-space it has be established that the *curvature of the flow*, i.e., the *curvature* of the *trajectory curves* of any n-dimensional dynamical system directly provides its *slow manifold* analytical equation the invariance of which has been proved according to *Darboux theorem*. Thus, it has been stated that since *curvature* only involves time derivatives of the velocity vector field and uses neither eigenvectors nor asymptotic expansions this simplifying method improves the *slow invariant manifold* analytical equation determination of high-dimensional dynamical systems. Chua's paradigmatic models and nonlinear magnetoconvection high-dimensional dynamical system have exemplified this result. Since it has been shown in the appendix that *curvature of the flow* generalizes the *Tangent Linear System Approximation* and encompasses the so-called *Geometric Singular Perturbation Theory*, it may be applied for *slow invariant manifolds* determination of various kinds of high-dimensional dynamical system such as *Chemical kinetics*, *Neuronal Bursting models*, *L.A.S.E.R. models*...

Two of the main perspectives to be given at this work may be highlighted. The former is *bifurcations*. It seems reasonable to consider that a *bifurcation* would modify the shape of the manifold and so conversely, geometric interpretations could enable to highlight such *bifurcations*. And the latter deals with the particular feature highlighted in Sec. 5.2, i.e., that *curvature of the flow* enables "detecting" *first integral* of dynamical systems. These works in progress will be developed in another publication.

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Appendix

The aim of this appendix is to present definitions inherent to *Differential Geometry* such as the concept of$n$-dimensional *smooth curves*, *generalized Frénet frame*, Gram-Schmidt orthogonalization process for computing *curvatures* of *trajectory curves* in Euclidean n-space as well as proofs of identities (A.10, A.15 & A.16) used in this work. Then, it is established that *curvature of the flow* for *slow invariant manifold* analytical equation determination of high-dimensional dynamical systems generalizes on the one hand the *tangent linear system approximation* [Rossetto *et al.*, 1998] and encompasses on the other hand the so-called *Geometric Singular Perturbation Theory* [Fenichel, 1979].

A Differential Geometry

Within the framework of *Differential Geometry*, n-dimensional *smooth curves*, i.e., *smooth curves* in Euclidean $n-$space are defined by a *regular parametric representation* in *terms of arc length* also called *natural representation* or *unit speed parametrization.* According to Herman Gluck [1966] local metrics properties of *curvatures* may be directly deduced from *curves parametrized in terms of time* and so *natural representation* is not necessary.

A.1 Concept of curves

Considering *trajectory curve* $\vec{X}(t)$ integral of a n-dimensional dynamical system (1) as "the motion of a variable point in a space of dimension n" leads to the following definition.

Définition A.1. *A smooth parametrized*[9] *curve in* $\mathbb{R}^n$ *is a smooth map* $\vec{X}(t) : [a,b] \to \mathbb{R}^n$ *from a closed interval* $[a,b]$ *into* $\mathbb{R}^n$. *A map is said to be smooth or infinitely many times differentiable if the coordinate functions* $x_1, x_2, \ldots, x_n$ *of* $\vec{X} = [x_1, x_2, \ldots, x_n]^t$ *have continuous partial derivatives of any order.*

A.2 Gram-Schmidt process and Frénet moving frame

There are many moving frames along a *trajectory curve* and most of them are not related to local metrics properties of *curvatures.* This is not the case for Frénet frame [1852]. In this sub-section generalized Frénet frame for n-dimensional *trajectory curves* in Euclidean n-space is recalled.

Let's suppose that the *trajectory curve* $\vec{X}(t)$, *parametrized in terms of time*, is of *general type* in $\mathbb{R}^n$, i.e., that the first $n-1$ time derivatives: $\dot{\vec{X}}(t)$, $\ddot{\vec{X}}(t)$, $\ldots$, $\overset{(n-1)}{\vec{X}}(t)$, are linearly independent for all t.

[9] with any kind of parametrization.

A moving frame along a *trajectory curve* $\vec{X}(t)$ of *general type* in $\mathbb{R}^n$ is a collection of i vectors $\vec{u}_1(t)$, $\vec{u}_2(t)$, ..., $\vec{u}_i(t)$ along $\vec{X}(t)$ forming an *orthogonal basis*, such that:

$$\vec{u}_i(t) \cdot \vec{u}_j(t) = 0 \tag{45}$$

for all t and for $i \neq j$. These vectors $\vec{u}_i(t)$ may be determined by application of the Gram-Schmidt orthogonalization process described below.

Gram-Schmidt process. Let $\dot{\vec{X}}(t)$, $\ddot{\vec{X}}(t)$, ..., $\overset{(n-1)}{\vec{X}}(t)$ be linearly independent vectors for all t in $\mathbb{R}^n$. According to Gram-Schmidt process [Lichnerowicz, 1950 p. 30, Gluck, 1966] the vectors $\vec{u}_1(t)$, $\vec{u}_2(t)$, ..., $\vec{u}_i(t)$ forming an orthogonal basis are defined by:

$$\vec{u}_1(t) = \dot{\vec{X}}(t)$$

$$\vec{u}_2(t) = \ddot{\vec{X}}(t) - \left(\frac{\vec{u}_1(t) \cdot \ddot{\vec{X}}(t)}{\vec{u}_1(t) \cdot \vec{u}_1(t)}\right)\vec{u}_1(t)$$

$$\vec{u}_3(t) = \dddot{\vec{X}}(t) - \left(\frac{\vec{u}_1(t) \cdot \dddot{\vec{X}}(t)}{\vec{u}_1(t) \cdot \vec{u}_1(t)}\right)\vec{u}_1(t) - \left(\frac{\vec{u}_2(t) \cdot \dddot{\vec{X}}(t)}{\vec{u}_2(t) \cdot \vec{u}_2(t)}\right)\vec{u}_2(t)$$

....

$$\vec{u}_i(t) = \overset{(i)}{\vec{X}}(t) - \sum_{j=1}^{n-1}\left(\frac{\vec{u}_j(t) \cdot \overset{(i)}{\vec{X}}(t)}{\vec{u}_j(t) \cdot \vec{u}_j(t)}\right)\vec{u}_j(t) \tag{46}$$

Generalized Frénet moving frame. Starting from the vectors $\vec{u}_1(t)$, $\vec{u}_2(t)$, ..., $\vec{u}_i(t)$ forming an orthogonal basis, *generalized Frénet moving frame* for the *trajectory curve* $\vec{X}(t)$ of *general type* in $\mathbb{R}^n$ may be built. Thus derivation with respect to time t leads to the *generalized Frénet formulas* in Euclidean n-space:

$$\dot{\vec{u}}_i(t) = v\sum_{j=1}^{n}\alpha_{ij}\vec{u}_j(t) \tag{47}$$

with $i = 1, 2, \ldots, n$ and where $v = \left\|\dot{\vec{X}}\right\| = \left\|\vec{V}\right\|$ represents the Euclidean norm of the velocity vector field. Moreover, according to Eq. (45) implies that:

$$\dot{\vec{u}}_i(t) \cdot \vec{u}_j(t) + \vec{u}_i(t) \cdot \dot{\vec{u}}_j(t) = 0 \tag{48}$$

So, $\alpha_{ii} = 0$ and $\alpha_{ij} = 0$ for $j < i - 1$. Thus, only $\alpha_{i,i+1} = -\alpha_{i+1,i}$ are not identically zero.

Let's pose:

$$\kappa_1 = \alpha_{12}, \quad \kappa_2 = \alpha_{23}, \quad \ldots, \quad \kappa_{n-1} = \alpha_{n-1,n} \tag{49}$$

The *generalized Frénet formulas* associated with a *trajectory curve* in Euclidean n-space read:

$$\left\{ \begin{array}{c} \dot{\vec{u}}_1(t) = v\kappa_1 \vec{u}_2(t) \\ \dot{\vec{u}}_2(t) = v\left[-\kappa_1 \vec{u}_1(t) + \kappa_2 \vec{u}_3(t)\right] \\ \dot{\vec{u}}_3(t) = -v\kappa_2 \vec{u}_2(t) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dot{\vec{u}}_{n-1}(t) = v\left[-\kappa_{n-2} \vec{u}_{n-2}(t) + \kappa_{n-1} \vec{u}_n(t)\right] \\ \dot{\vec{u}}_n(t) = -v\kappa_{n-1} \vec{u}_{n-1}(t) \end{array} \right. \tag{50}$$

The functions κ_1, κ_2, $\ldots$, κ_{n-1} are called *curvatures* of *trajectory curve* $\vec{X}(t)$ of *general type* in $\mathbb{R}^n$ and κ_{n-1} is analogous to the *torsion*.

Thus, according to Gluck [1966, p. 702] *curvatures* of *trajectory curves* $\vec{X}(t)$ integral of any n-dimensional dynamical systems (1) may be defined by:

$$\kappa_i = \frac{\left\| \vec{u}_{i+1}(t) \right\|}{\left\| \vec{u}_1(t) \right\| \left\| \vec{u}_i(t) \right\|} \tag{51}$$

Since $1 \leqslant i \leqslant n-1$ a n-dimensional *trajectory curve* has $(n-1)$ *curvatures*.

A.3 Frénet trihedron and curvatures of space curves

Frénet trihedron. While normalizing the basis vectors $\vec{u}_1(t)$, $\vec{u}_2(t)$, $\ldots$, $\vec{u}_n(t)$ obtained with the Gram-Schmidt process, the so-called Frénet trihedron for *space curves* may be deduced.
Hence, it may be stated that: $\left(\frac{\vec{u}_1(t)}{\|\vec{u}_1(t)\|}, \frac{\vec{u}_2(t)}{\|\vec{u}_2(t)\|}, \frac{\vec{u}_3(t)}{\|\vec{u}_3(t)\|} \right) = \left(\vec{\tau}, \vec{n}, \vec{b} \right)$ where $\vec{\tau}$, $\vec{n}$ and $\vec{b}$ are respectively the tangent, normal and binormal unit vectors.

Let's notice that the three first time derivatives: $\dot{\vec{X}}(t)$, $\ddot{\vec{X}}(t)$ and $\dddot{\vec{X}}(t)$ represent respectively the velocity, acceleration and over-acceleration vector field namely: $\vec{V}(t)$, $\vec{\gamma}(t)$ and $\dot{\vec{\gamma}}(t)$. Thus, from the *generalized Frénet formulas* (50) and Gluck formulae (51) of *curvatures*, the first and second curvatures of *space curves*, i.e., *curvature* and *torsion* may be found again.

First curvature. While replacing basis vectors $\vec{u}_1(t)$ and $\vec{u}_2(t)$ resulting from the Gram-Schmidt process in formulae (51), (first) *curvature* of *space trajectory curves* is given by:

$$\kappa_1(t) = \frac{\left\| \vec{u}_2(t) \right\|}{\left\| \vec{u}_1(t) \right\|^2} = \frac{\left\| \vec{\gamma}(t) \wedge \vec{V}(t) \right\|}{\left\| \vec{V} \right\|^3} \tag{52}$$

Proof. While using the Lagrange identity it may be established that:

$\|\vec{u}_1\|^2 \|\vec{u}_2\|^2 = \left\| \dot{\vec{X}} \wedge \ddot{\vec{X}} \right\|^2$. So, *curvature* κ_1 reads:

$$\kappa_1(t) = \frac{\|\vec{u}_2(t)\|}{\|\vec{u}_1(t)\|^2} = \frac{\left\| \vec{\gamma}(t) \wedge \vec{V}(t) \right\|}{\left\| \vec{V} \right\|^3}$$

□

Second curvature. While replacing basis vectors $\vec{u}_1(t)$, $\vec{u}_2(t)$ and $\vec{u}_3(t)$ resulting from the Gram-Schmidt process in formulae (51), (second) *curvature*, i.e., *torsion* of *space trajectory curves* is given by:

$$\kappa_2(t) = \frac{\|\vec{u}_3(t)\|}{\|\vec{u}_1(t)\| \|\vec{u}_2(t)\|} = -\frac{\dot{\vec{\gamma}}(t) \cdot \left(\vec{\gamma}(t) \wedge \vec{V}(t) \right)}{\left\| \vec{\gamma}(t) \wedge \vec{V}(t) \right\|^2} \tag{53}$$

Proof. Still using the Lagrange identity, i.e., $\|\vec{u}_1\|^2 \|\vec{u}_2\|^2 = \left\| \dot{\vec{X}} \wedge \ddot{\vec{X}} \right\|^2$ *torsion* κ_2 reads:

$$\kappa_2 = \frac{\|\vec{u}_3(t)\|}{\|\vec{u}_1(t)\| \|\vec{u}_2(t)\|} = -\frac{\dot{\vec{\gamma}} \cdot \left(\vec{\gamma} \wedge \vec{V} \right)}{\left\| \vec{\gamma} \wedge \vec{V} \right\|^2}$$

□

A.4 Identities proofs

Identity A.10.

$$\left[\dot{\vec{X}}, \ddot{\vec{X}}, \ldots, \overset{(n)}{\vec{X}} \right] = \dot{\vec{X}} \cdot \left(\ddot{\vec{X}} \wedge \dddot{\vec{X}} \wedge \ldots \wedge \overset{(n)}{\vec{X}} \right) = \|\vec{u}_1\| \|\vec{u}_2\| \ldots \|\vec{u}_n\| \tag{54}$$

Proof. According to Postnikov [1981, p. 215], Gram-Schmidt process can be written

$$\vec{u}_n(t) = \sum_{i=1}^{n} \beta_{ni} \overset{(i)}{\vec{X}}(t) \tag{55}$$

Comparing (55) with (46) leads to:

$$\beta_{ii} = 1 \tag{56}$$

Using (55) and (56), the *inner product* $\vec{u}_1 \cdot (\vec{u}_2 \wedge \ldots \wedge \vec{u}_n)$ reads:

$$\vec{u}_1 \cdot (\vec{u}_2 \wedge \ldots \wedge \vec{u}_n) = \dot{\vec{X}} \cdot \left(\ddot{\vec{X}}, \ldots, \overset{(n)}{\vec{X}} \right) \tag{57}$$

But, since Gram-Schmidt basis is orthogonal, the *inner product* $\vec{u}_1 \cdot (\vec{u}_2 \wedge \ldots \wedge \vec{u}_n)$ reads too:

$$\vec{u}_1 \cdot (\vec{u}_2 \wedge \ldots \wedge \vec{u}_n) = \|\vec{u}_1\| \, \|\vec{u}_2\| \cdots \|\vec{u}_n\| \tag{58}$$

From (57) and (58) it follows that: $\dot{\vec{X}} \cdot \left(\ddot{\vec{X}}, \ldots, \overset{(n)}{\vec{X}} \right) = \|\vec{u}_1\| \, \|\vec{u}_2\| \cdots \|\vec{u}_n\|$.

For example, while omitting the *time* variable the three first Gram-Schmidt vectors read:

$\vec{u}_1 = \beta_{11} \dot{\vec{X}}$
$\vec{u}_2 = \beta_{21} \dot{\vec{X}} + \beta_{22} \ddot{\vec{X}}$
$\vec{u}_3 = \beta_{31} \dot{\vec{X}} + \beta_{32} \ddot{\vec{X}} + \beta_{33} \dddot{\vec{X}}$

Using (55) and (56), the *inner product* $\vec{u}_1 \cdot (\vec{u}_2 \wedge \vec{u}_3)$ reads:

$$\vec{u}_1 \cdot (\vec{u}_2 \wedge \vec{u}_3) = \beta_{11} \beta_{22} \beta_{33} \dot{\vec{X}} \cdot \left(\ddot{\vec{X}} \wedge \dddot{\vec{X}} \right) = \|\vec{u}_1\| \, \|\vec{u}_2\| \, \|\vec{u}_3\|$$

□

Identity A.15.

$$J\vec{a}_1 . (J\vec{a}_2 \wedge \ldots \wedge J\vec{a}_n) = Det\,(J)\, \vec{a}_1 . (\vec{a}_2 \wedge \ldots \wedge \vec{a}_n) \tag{59}$$

Proof. Equation (59) may also be written with *inner product*:

$$J\vec{a}_1 . (J\vec{a}_2 \wedge \ldots \wedge J\vec{a}_n) = [J\vec{a}_1, J\vec{a}_2, \ldots, J\vec{a}_n] = Det\,(J\vec{a}_1, J\vec{a}_2, \ldots, J\vec{a}_n)$$

But, since $(J\vec{a}_1, J\vec{a}_2, \ldots, J\vec{a}_n) = J\,(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n)$ and while using determinant product property, i.e., determinant of the product is equal to the product of the determinants we have:

$$\begin{aligned} J\vec{a}_1.\left(J\vec{a}_2 \wedge \ldots \wedge J\vec{a}_n\right) &= \left[J\vec{a}_1, J\vec{a}_2, \ldots, J\vec{a}_n\right] \\ &= Det\left(J\vec{a}_1, J\vec{a}_2, \ldots, J\vec{a}_n\right) \\ &= Det\left(J\right) Det\left(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\right) \end{aligned} \tag{60}$$

□

Identity A.16.

$$\begin{aligned} J\vec{a}_1.\left(\vec{a}_2 \wedge \ldots \wedge \vec{a}_n\right) + \vec{a}_1.\left(J\vec{a}_2 \wedge \ldots \wedge \vec{a}_n\right) + \quad \ldots \\ + \vec{a}_1.\left(\vec{a}_2 \wedge \ldots \wedge J\vec{a}_n\right) = Tr\left(J\right) \vec{a}_1.\left(\vec{a}_2 \wedge \ldots \wedge \vec{a}_n\right) \end{aligned} \tag{61}$$

Proof. The proof is based on Trace properties such as *linearity* and *similarity-invariant*.

□

B Tangent linear system approximation

B.1 Assumptions

The *generalized tangent linear system approximation* requires that the dynamical system (1) satisfies the following assumptions:

(H$_1$) The components f_i, of the velocity vector field $\vec{\Im}\left(\vec{X}\right)$ defined in E are continuous, C^∞ functions in E and with values included in $\mathbb{R}$.

(H$_2$) The dynamical system (1) satisfies the *nonlinear part condition* [Rossetto *et al.*, 1998], i.e., that the influence of the nonlinear part of the Taylor series of the velocity vector field $\vec{\Im}\left(\vec{X}\right)$ of this system is overshadowed by the fast dynamics of the linear part.

$$\vec{\Im}\left(\vec{X}\right) = \vec{\Im}\left(\vec{X}_0\right) + \left(\vec{X} - \vec{X}_0\right) \left.\frac{d\vec{\Im}\left(\vec{X}\right)}{d\vec{X}}\right|_{\vec{X}_0} + O\left(\left(\vec{X} - \vec{X}_0\right)^2\right) \tag{62}$$

(H$_3$) The functional jacobian matrix associated to dynamical system (1) has at least a "fast" eigenvalue λ_1, i.e., with the largest absolute value of the real part.

B.2 Corollaries

To the dynamical system (1) is associated a *tangent linear system* defined as follows:

$$\frac{d\delta\vec{X}}{dt} = J\left(\vec{X}_0\right)\delta\vec{X} \tag{63}$$

where

$\delta\vec{X} = \vec{X} - \vec{X}_0, \quad \vec{X}_0 = \vec{X}\left(t_0\right)$and $\left.\frac{d\vec{\Im}\left(\vec{X}\right)}{d\vec{X}}\right|_{\vec{X}_0} = J\left(\vec{X}_0\right)$

Corollary B.1.

The *nonlinear part condition* implies that the velocity varies slowly in the vicinity of the *slow manifold*. This involves that the functional jacobian $J\left(\vec{X}_0\right)$ varies slowly with time, i.e.,

$$\frac{dJ}{dt}\left(\vec{X}_0\right) = 0 \tag{64}$$

The solution of the *tangent linear system* (63) is written:

$$\delta\vec{X} = e^{J\left(\vec{X}_0\right)(t-t_0)}\delta\vec{X}\left(t_0\right) \tag{65}$$

So,

$$\delta\vec{X} = \sum_{i=1}^{n} a_i \overrightarrow{Y_{\lambda_i}} \tag{66}$$

where n is the dimension of the eigenspace, a_i represents coefficients depending explicitly on the co-ordinates of space and implicitly on time and $\overrightarrow{Y_{\lambda_i}}$ the eigenvectors associated in the functional jacobian of the *tangent linear system*.

Corollary B.2.

In the vicinity of the *slow manifold* the velocity of the dynamical system (1) and that of the *tangent linear system* (63) merge.

$$\frac{d\delta\vec{X}}{dt} = \vec{V}_T \approx \vec{V} \tag{67}$$

where $\vec{V}_T$ represents the velocity vector associated with the *tangent linear system*.

The *tangent linear system approximation* consists in spreading the velocity vector field $\vec{V}$ on the eigenbasis associated to the functional jacobian matrix of the *tangent linear system*.

While taking account of (63) and (66) we have according to (67):

$$\frac{d\delta\vec{X}}{dt} = J\left(\vec{X}_0\right)\delta\vec{X} = J\left(\vec{X}_0\right)\sum_{i=1}^{n} a_i\overrightarrow{Y_{\lambda_i}} = \sum_{i=1}^{n} a_i J\left(\vec{X}_0\right)\overrightarrow{Y_{\lambda_i}} = \sum_{i=1}^{n} a_i\lambda_i\overrightarrow{Y_{\lambda_i}} \tag{68}$$

Thus, Corollary B.2 provides:

$$\frac{d\delta\vec{X}}{dt} = \overrightarrow{V}_T \approx \overrightarrow{V} = \sum_{i=1}^{n} a_i \lambda_i \overrightarrow{Y_{\lambda_i}} \tag{69}$$

Then, existence of an evanescent mode in the vicinity of the *slow manifold* implies according to Tikhonov's theorem [1952] that $a_1\lambda_1 \ll 1$. So, the *coplanarity* condition (69) provides the *slow manifold* equation of a n-dimensional dynamical system (1).

Proposition B.1. *The coplanarity condition between the velocity vector field $\overrightarrow{V}$ of a n-dimensional dynamical system and the slow eigenvectors $\overrightarrow{Y_{\lambda_i}}$ associated to the slow eigenvalues λ_i of its functional jacobian provides the slow manifold equation of such system.*

$$\overrightarrow{V} = \sum_{i=2}^{n} a_i \overrightarrow{Y_{\lambda_i}} = a_2\overrightarrow{Y_{\lambda_2}} + \ldots + a_n\overrightarrow{Y_{\lambda_n}} \quad \Leftrightarrow \quad \phi\left(\vec{X}\right) = \overrightarrow{V} . \left(\overrightarrow{Y_{\lambda_2}} \wedge \ldots \wedge \overrightarrow{Y_{\lambda_n}}\right) = 0 \tag{70}$$

An alternative proposed by Rossetto *et al.* [1998] uses the "fast" eigenvector on the left associated with the "fast" eigenvalue of the transposed functional jacobian of the *tangent linear system*. In this case the velocity vector field $\overrightarrow{V}$ is then orthogonal with the "fast" eigenvector on the left. This *orthogonality* condition also provides the *slow manifold* equation of a n-dimensional dynamical system (1).

Proposition B.2. *The orthogonality condition between the velocity vector field $\overrightarrow{V}$ of a n-dimensional dynamical system and the fast eigenvector ${}^t\overrightarrow{Y_{\lambda_1}}$ on the left associated with the fast eigenvalue λ_1 of its transposed functional jacobian provides the slow manifold equation of such system.*

$$\phi\left(\vec{X}\right) = \overrightarrow{V} \cdot {}^t\overrightarrow{Y_{\lambda_1}} = 0 \tag{71}$$

Proposition B.3. *Both coplanarity and orthogonality conditions providing the slow manifold equation are equivalent.*

While using the following identity the proof is obvious:

$$\left(\overrightarrow{Y_{\lambda_2}} \wedge \overrightarrow{Y_{\lambda_3}} \wedge \ldots \wedge \overrightarrow{Y_{\lambda_n}}\right) = {}^t\overrightarrow{Y_{\lambda_1}} \tag{72}$$

Thus, *coplanarity* and *orthogonality* conditions are completely equivalent.

Since for low-dimensional two and three dynamical systems the proof has been already established [Ginoux *et al.*, 2006] while using the *Tangent Linear System Approximation*, for high-dimensional dynamical systems it may be deduced from its generalization presented above. Thus, according to the generalization of the *Tangent Linear System Approximation* the *slow manifold* equation of a n-dimensional dynamical system may be written:

$$\phi\left(\vec{X}\right) = \vec{V}.\left(\overrightarrow{Y_{\lambda_2}} \wedge \ldots \wedge \overrightarrow{Y_{\lambda_n}}\right) = 0 \quad \Leftrightarrow \quad \vec{V} = \sum_{i=2}^{n} a_i \overrightarrow{Y_{\lambda_i}} = a_2 \overrightarrow{Y_{\lambda_2}} + \ldots + a_n \overrightarrow{Y_{\lambda_n}} \tag{73}$$

In the framework of the *Generalized Tangent Linear System Approximation* the functional jacobian matrix associated to the dynamical system has been supposed to be stationary:

$$\frac{dJ}{dt} = 0 \tag{74}$$

As a consequence, time derivatives of acceleration vectors reads:

$$\overset{(n)}{\vec{\gamma}} = J^{(n+1)} \vec{V} = J^{(n)} \vec{\gamma}$$

Then, mapping the *flow* of the *tangent linear system*, i.e., functional jacobian operator J to the velocity vector field spanned on the eigenbasis (73) leads to:

$$J\vec{V} = \vec{\gamma} = \sum_{i=2}^{n} a_i J\overrightarrow{Y_{\lambda_i}} = a_2 J\overrightarrow{Y_{\lambda_2}} + \ldots + a_n J\overrightarrow{Y_{\lambda_n}}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$J^{(n-2)} \vec{V} = \overset{(n-2)}{\vec{\gamma}} = \sum_{i=2}^{n} a_i J^{(n-2)} \overrightarrow{Y_{\lambda_i}} = a_2 J^{(n-2)} \overrightarrow{Y_{\lambda_2}} + \ldots + a_n J^{(n-2)} \overrightarrow{Y_{\lambda_n}}$$

While using the eigenequation: $J\overrightarrow{Y_{\lambda_k}} = \lambda_k \overrightarrow{Y_{\lambda_k}}$ these equations read:

$$J\vec{V} = \vec{\gamma} = \sum_{i=2}^{n} a_i \lambda_i \overrightarrow{Y_{\lambda_i}} = a_2 \lambda_2 \overrightarrow{Y_{\lambda_2}} + \ldots + a_n \lambda_n \overrightarrow{Y_{\lambda_n}}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$J^{(n-2)} \vec{V} = \overset{(n-2)}{\vec{\gamma}} = \sum_{i=2}^{n} a_i J^{(n-2)} \overrightarrow{Y_{\lambda_i}} = a_2 \lambda_2^{n-2} \overrightarrow{Y_{\lambda_2}} + \ldots + a_n \lambda_n^{n-2} \overrightarrow{Y_{\lambda_n}}$$

Under the assumptions of the *Tangent Linear System Approximation*, it is obvious that the vectors $\vec{V}, \vec{\gamma}, \ \ldots, \ \overset{(n-2)}{\vec{\gamma}}$ spanned on the same eigenbasis $\left(\overrightarrow{Y_{\lambda_2}}, \overrightarrow{Y_{\lambda_3}}, \ldots, \overrightarrow{Y_{\lambda_n}}\right)$ are "hypercoplanar". This implies that

$$\vec{V} \cdot \left(\vec{\gamma} \wedge \dot{\vec{\gamma}} \wedge \ldots \wedge \overset{(n-2)}{\vec{\gamma}}\right) = 0 \quad \Leftrightarrow \quad \dot{\vec{X}} \cdot \left(\ddot{\vec{X}} \wedge \dddot{\vec{X}} \wedge \ldots \wedge \overset{(n)}{\vec{X}}\right) = 0$$

Thus, *curvature of the flow* generalizes and encompasses *Tangent Linear System Approximation.*

□

C Geometric Singular Perturbation Theory

Dynamical systems (1) with small multiplicative parameters in one or several components of their velocity vector field, i.e., *singularly perturbed systems* may be defined as:

$$\left\{ \begin{array}{l} \vec{x}' = \vec{f}\left(\vec{x}, \vec{z}, \varepsilon\right) \\ \vec{z}' = \varepsilon \vec{g}\left(\vec{x}, \vec{z}, \varepsilon\right) \end{array} \right. \tag{75}$$

where $\vec{x} \in \mathbb{R}^m$, $\vec{z} \in \mathbb{R}^n$, $\varepsilon \in \mathbb{R}^+$ and the prime denotes differentiation with respect to the independent variable t. The functions $\vec{f}$ and $\vec{g}$ are assumed to be C^∞ functions of $\vec{x}$, $\vec{z}$ and ε in $U \times I$, where U is an open subset of $\mathbb{R}^m \times \mathbb{R}^n$ and I is an open interval containing $\varepsilon = 0$. When $\varepsilon \ll 1$, i.e., is a small positive number, the variable $\vec{x}$ is called *fast* variable, and $\vec{z}$ is called *slow* variable. Using Landau's notation: $O\left(\varepsilon^k\right)$ represents a real polynomial in ε of k degree, with $k \in \mathbb{Z}$, it is used to consider that generally $\vec{x}$ evolves at an $O\left(1\right)$ rate; while $\vec{z}$ evolves at an $O\left(\varepsilon\right)$ *slow* rate. Reformulating the system (1) in terms of the rescaled variable $\tau = \varepsilon t$, we obtain:

$$\left\{ \begin{array}{l} \varepsilon \dot{\vec{x}} = \vec{f}\left(\vec{x}, \vec{z}, \varepsilon\right) \\ \dot{\vec{z}} = \vec{g}\left(\vec{x}, \vec{z}, \varepsilon\right) \end{array} \right. \tag{76}$$

The dot $(\cdot)$ represents as the derivative with respect to the new independent variable τ. The independent variables t and τ are referred to the *fast* and *slow* times, respectively, and (75) and (76) are called *fast* and *slow* system, respectively. These systems are equivalent whenever $\varepsilon \neq 0$, and they are labelled *singular perturbation problems* when $\varepsilon \ll 1$, i.e., is a small positive parameter. The label *singular* stems in part from the discontinuous limiting behaviour in the system (75) as $\varepsilon \to 0^+$. In such case, the system (75) reduces to an m-dimensional system called *reduced fast system*, with the variable $\vec{z}$ as a constant parameter. System (76) leads to a differential-algebraic system called *reduced slow system* which dimension decreases from $m+n$ to n.By exploiting the decomposition into *fast* and *slow reduced systems* the geometric approach reduced the full *singularly perturbed system* to separate lower-dimensional regular perturbation problems in the *fast* and *slow* regimes, respectively. *Geometric Singular Perturbation Theory* is based on Fenichel's assumptions [Fenichel, 1979] recalled below.

C.1 Assumptions

(H$_1$) The functions $\vec{f}$ and $\vec{g}$ are C^{∞} functions in $U \times I$, where U is an open subset of $\mathbb{R}^m \times \mathbb{R}^n$ and I is an open interval containing $\varepsilon = 0$.

(H$_2$) There exists a set M_0 that is contained in $\{(\vec{x}, \vec{z}) : \vec{f}(\vec{x}, \vec{z}, 0) = 0\}$ such that M_0 is a compact manifold with boundary and M_0 is given by the graph of a C^1 function $\vec{x} = \vec{X}_0(\vec{z})$ for $\vec{z} \in D$, where $D \subseteq \mathbb{R}^n$ is a compact, simply connected domain and the boundary of D is an $(n-1)$ dimensional C^{∞} submanifold. Finally, the set D is overflowing invariant with respect to (76) when $\varepsilon = 0$.

(H$_2$) M_0 is normally hyperbolic relative to the *reduced fast system* and in particular it is required for all points $\vec{p} \in M_0$, that there are k (resp. l) eigenvalues of $D_{\vec{x}}\vec{f}(\vec{p}, 0)$ with positive (resp. negative) real parts bounded away from zero, where $k + l = m$.

C.2 Theorems

Fenichel's persistence theorem.

Let system (75) satisfying the conditions $(H_1) - -(H_3)$. If $\varepsilon > 0$ is sufficiently small, then there exists a function $\vec{X}(\vec{z}, \varepsilon)$ defined on D such that the manifold $M_{\varepsilon} = \{(\vec{x}, \vec{z}) : \vec{x} = \vec{X}(\vec{z}, \varepsilon)\}$ is locally invariant under (75). Moreover, $\vec{X}(\vec{z}, \varepsilon)$ is C^r for any $r < +\infty$, and M_{ε} is $C^r O(\varepsilon)$ close to M_0. In addition, there exist perturbed local stable and unstable manifolds of M_{ε}. They are unions of invariant families of stable and unstable fibers of dimensions l and k, respectively, and they are $C^r O(\varepsilon)$ close for all $r < +\infty$, to their counterparts.

Invariance.

Generally, Fenichel theory enables to turn the problem for explicitly finding functions $\vec{x} = \vec{X}(\vec{z}, \varepsilon)$ whose graphs are locally *slow invariant manifolds* M_{ε} of system (75) into regular perturbation problem. Invariance of the manifold M_{ε} implies that $\vec{X}(\vec{z}, \varepsilon)$ satisfies:

$$\varepsilon D_{\vec{z}}\vec{X}(\vec{z}, \varepsilon)\, \vec{g}\left(\vec{X}(\vec{z}, \varepsilon), \vec{z}, \varepsilon\right) = \vec{f}\left(\vec{X}(\vec{z}, \varepsilon), \vec{z}, \varepsilon\right) \tag{77}$$

Then, the following perturbation expansion is plugged: $\vec{X}(\vec{z}, \varepsilon) = \vec{X}_0(\vec{z}) + \varepsilon \vec{X}_1(\vec{z}) + O\left(\varepsilon^2\right)$ into (77) to solve order by order for $\vec{X}(\vec{z}, \varepsilon)$. The Taylor series expansion for $\vec{f}\left(\vec{X}(\vec{z}, \varepsilon), \vec{z}, \varepsilon\right)$ up to terms of order two in ε leads at order $\varepsilon^0$to

$$\vec{f}\left(\vec{X}_0(\vec{z}, \varepsilon), \vec{z}, 0\right) = \vec{0} \tag{78}$$

which defines $\vec{X}_0(\vec{z})$ due to the invertibility of $D_{\vec{x}}\vec{f}$ and the *implicit function theorem*. At order ε^1 we have:

$$D_{\vec{z}}\vec{X}_0(\vec{z})\,\vec{g}\left(\vec{X}_0(\vec{z}),\vec{z},0\right) = D_{\vec{x}}\vec{f}\left(\vec{X}_0(\vec{z}),\vec{z},0\right)\vec{X}_1(\vec{z}) + \frac{\partial\vec{f}}{\partial\varepsilon}\left(\vec{X}_0(\vec{z}),\vec{z},0\right) \quad (79)$$

which yields $\vec{X}_1(\vec{z})$ and so forth.

$$D_{\vec{x}}\vec{f}\left(\vec{X}_0(\vec{z}),\vec{z},0\right)\vec{X}_1(\vec{z}) = D_{\vec{z}}\vec{X}_0(\vec{z})\,\vec{g}\left(\vec{X}_0(\vec{z}),\vec{z},0\right) - \frac{\partial\vec{f}}{\partial\varepsilon}\left(\vec{X}_0(\vec{z}),\vec{z},0\right) \quad (80)$$

So, regular perturbation theory enables to build locally *slow invariant manifolds* M_ε. But for high-dimensional *singularly perturbed systems slow invariant manifold* analytical equation determination leads to tedious calculations.

Let's write the *slow invariant manifold* (2) defined by the *curvature of the flow* as:

$$\phi(\vec{x},\vec{z},\varepsilon) = 0 \quad (81)$$

Proof. Plugging the perturbation expansion: $\vec{X}(\vec{z},\varepsilon) = \vec{X}_0(\vec{z}) + \varepsilon\vec{X}_1(\vec{z}) + O\left(\varepsilon^2\right)$ into Eq. (81) to solve order by order for $\vec{X}(\vec{z},\varepsilon)$. The Taylor series expansion for $\phi\left(\vec{X}(\vec{z},\varepsilon),\vec{z},\varepsilon\right)$ up to terms of suitable order in ε leads to the same coefficients as those obtained above.

Order ε^0 provides:

$$\phi\left(\vec{X}_0(\vec{z},\varepsilon),\vec{z},0\right) = 0$$

which also defines $\vec{X}_0(\vec{z})$ due to the invertibility of $D_{\vec{x}}\vec{f}$ and the *implicit function theorem*. Thus, *curvature of the flow* encompasses *Geometric Singular Perturbation Theory*.

□