# Invariant Manifolds of Complex Systems

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**Abstract**

The aim of this work is to establish the existence of invariant manifolds in *complex systems*. Considering *trajectory curves* integral of multiple time scales dynamical systems of dimension two and three (predator-prey models, neuronal bursting models) it is shown that there exists in the phase space a *curve* (resp. a *surface*) which is invariant with respect to the flow of such systems. These invariant manifolds are playing a very important role in the stability of complex systems in the sense that they are "restoring" the determinism of *trajectory curves*.

## 1 Dynamical systems

In the following we consider a system of ordinary differential equations defined in a compact E included in

$$\frac{d\vec{X}}{dt} = \vec{\Im}\left(\vec{X}\right) \tag{1}$$

with $\vec{X} = [x_1, x_2, ..., x_n]^t \in E \subset \mathbb{R}^n$ and

$$\vec{\Im}\left(\vec{X}\right) = \left[f_1\left(\vec{X}\right), f_2\left(\vec{X}\right), ...f_n\left(\vec{X}\right)\right]^t \in E \subset \mathbb{R}^n .$$

The vector $\vec{\Im}\left(\vec{X}\right)$ defines a velocity vector field in E whose components $f_1$ which are supposed to be continuous and infinitely differentiable with respect to all $x_t$ and $t$, i.e., are $C^\infty$ functions in E and with values included in $\mathbb{R}$. For more details, see for example [1]. A solution of this system is an *integral curve* $\vec{X}(t)$ tangent to $\vec{\Im}$ whose values define the states of the dynamical system described by Equation (1). Since none of the components $f_i$ of the velocity vector field depends here explicitly on time, the system is said to be autonomous.

# 2 Trajectory curves

The integral of the system (1) can be associated with the coordinates, i.e., with the position, of a point M at the instant $t$. The total derivative of $\vec{V}(t)$ namely the instantaneous acceleration vector $\vec{\gamma}(t)$ may be written, while using the chain rule, as:

$$\vec{\gamma} = \frac{\mathrm{d}\vec{V}}{\mathrm{d}t} = \frac{\mathrm{d}\vec{\Im}}{\mathrm{d}\vec{X}}\frac{\mathrm{d}\vec{X}}{\mathrm{d}t} = J\vec{V} \tag{2}$$

where $\frac{\mathrm{d}\vec{\Im}}{\mathrm{d}\vec{X}}$ is the functional jacobian matrix $J$ of the system (1). Then, the *integral curve* defined by the vector function $\vec{X}(t)$ of the scalar variable $t$ representing the trajectory of M can be considered as a *plane* or a *space curve* which has local metrics properties namely *curvature* and *torsion*.

## 2.1 Curvature

The curvature, which expresses the rate of changes of the tangent to the trajectory curve of system (1), is defined by

$$\frac{1}{\Re} = \frac{\|\vec{\gamma} \wedge \vec{V}\|}{\|\vec{V}\|^3} \tag{3}$$

where $\Re$ represents the *radius of curvature*.

## 2.2 Torsion

The *torsion*, which expresses the difference between the *trajectory curve* of system (1) and a *plane curve*, is defined by:

$$\frac{1}{\Im} = -\frac{\dot{\vec{\gamma}} \cdot \left(\vec{\gamma} \wedge \vec{V}\right)}{\|\vec{\gamma} \wedge \vec{V}\|^2} \tag{4}$$

where $\Im$ represents the *radius of torsion*.

## 3 Lie Derivative — Darboux Invariant

Let $\varphi$ a $C^4$ function defined in a compact E included in $\mathbb{R}$ and $\vec{X}(t)$ the integral of the dynamic system defined by (1). The Lie derivative is defined as follows:

$$L_{\vec{X}}\varphi = \vec{V} \cdot \vec{\nabla}\varphi = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}\dot{x}_i = \frac{\mathrm{d}\varphi}{\mathrm{d}t} \tag{5}$$

**Theorem 1.** *An invariant curve (resp. surface) is defined by* $\varphi(\vec{X}) = 0$ *where* $\varphi$ *is a* $C^1$ *in an open set* $U$ *and such there exists a* $C^4$ *function denoted* $k(\vec{X})$ *and called cofactor which satisfies*

$$L_{\vec{X}}\phi(\vec{X}) = k(\vec{X})\phi(\vec{X}) \tag{6}$$

*for all* $\vec{X} \in U$.

Proof of this theorem may be found in [2].

**Theorem 2.** *If* $L_{\vec{X}}\varphi = 0$ *then* $\varphi$ *is first integral of the dynamical system defined by (1). So,* $\varphi$ *is first integral of the dynamical system defined by* $\{\varphi = \alpha\}$ *and where* $\alpha$ *is constant.*

Proof of this theorem may be found in [3].

## 4 Invariant Manifolds

According to the previous theorems 1 and 2 the following proposition may be established.

**Proposition 1.** *The location of the points where the local curvature of the trajectory curves integral of a two dimensional dynamical system defined by (1) vanishes is first integral of this system. Moreover, the invariant curve thus defined is over flowing invariant with respect to the dynamical system (1).*

Proof of this theorem may be found in [5].

**Proposition 2.** *The location of the points where the local torsion of the trajectory curves integral of a three dimensional dynamical system defined by (1) vanishes is first integral of this system. Moreover, the invariant surface thus defined is over flowing invariant with respect to the dynamical system (1).*

Proof of this theorem may be found in [5].

# 5 Applications to Complex Systems

According to this method it is possible to show that any dynamical system defined by (1) possess an invariant manifold which is endowing stability with the trajectory curves, restoring thus the loss determinism inherent to the non-integrability feature of these systems. So, this method may be also applied to any complex system such that predator-prey models, neuronal bursting models... But, in order to give the most simple and consistent application, let's focus on two classical examples:

- the Balthazar Van der Pol model;
- the Lorenz model.

## 5.1 Van der Pol model

The oscillator of B. Van der Pol [7] is a second-order system with non-linear frictions which can be written:

$$\ddot{x} + \alpha\left(x^2 - 1\right)\dot{x} + x = 0\,.$$

The particular form of the friction which can be carried out by an electric circuit causes a decrease of the amplitude of the great oscillations and an increase of the small. There are various manners of writing the previous equation like a first order system. One of them is:

$$\begin{cases} \dot{x} = \alpha\left(x + y - \dfrac{x^3}{3}\right) \\ \dot{y} = -\dfrac{x}{\alpha} \end{cases}$$

When $\alpha$ becomes very large, $x$ becomes a fast variable and $y$ a slow variable. In order to analyze the limit $\alpha \to \infty$, we introduce a small parameter $\epsilon = \frac{1}{\alpha^2}$ and a slow time $t' = \frac{t}{\alpha}\sqrt{\epsilon t}$. Thus, the system can be written:

$$\vec{V} = \begin{pmatrix} \dfrac{\mathrm{d}x}{\mathrm{d}t} \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} \end{pmatrix} = \Im \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\epsilon}\left(x + y - \dfrac{x^3}{3}\right) \\ -x \end{pmatrix} \tag{7}$$

with $\epsilon$ a positive real parameter: $\epsilon = 0.05$ and where the functions $f$ and $g$ are infinitely differentiable with respect to all $x_i$ and $t$, i.e. are $C^\infty$ functions in a compact E included in $\mathbb{R}^2$ and with values in $\mathbb{R}$.

According to Proposition 1, the location of the points where the local *curvature* vanishes leads to the following equation:

$$\phi(x,y) = 9y^2 + \left(9x + 3x^3\right)y + 6x^4 - 2x^6 + 9x^2\epsilon \tag{8}$$

According to Theorem 1 (Cf. Appendix for details), the Lie derivative of Equation (8) may be written:

$$L_{\vec{X}}\phi(\vec{X}) = \mathrm{Tr}[J]\phi(\vec{X}) + \frac{2x^2}{\epsilon}\left(-3x - 3y + x^3\right) \tag{9}$$

Let's plot the function $\phi(x,y)$ (in blue), its Lie derivative $L_{\vec{X}}\phi(\vec{X})$ (in magenta), the *singular approximation* $x + y - \frac{x^3}{3}$ (in green) and the *limit cycle* corresponding to system (7) (in red):

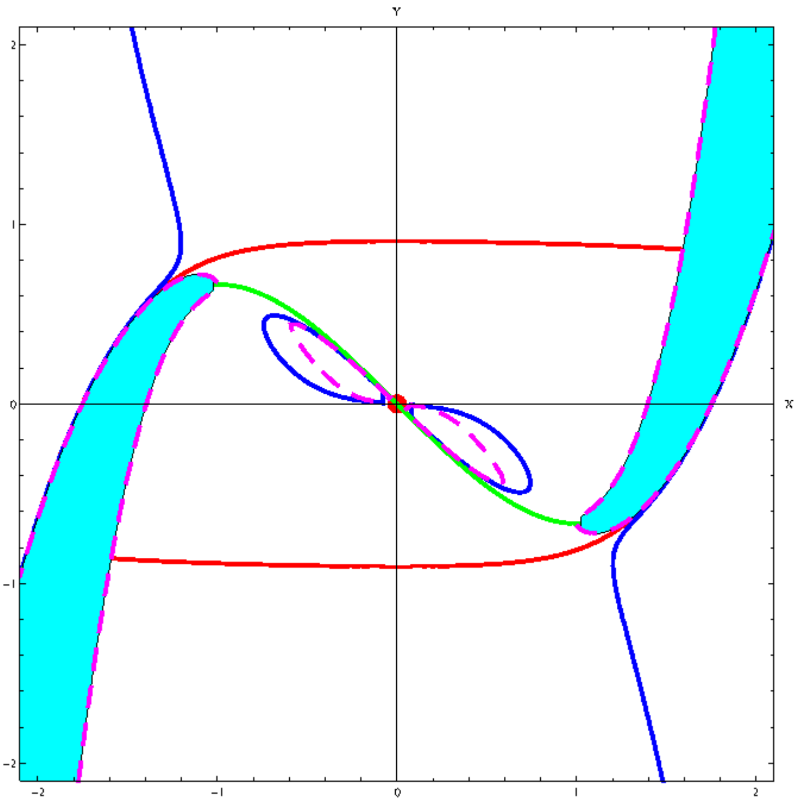


Figure 1: Van der Pol model.

According to Fenichel's theory, there exists a function $\varphi(x, y)$ defining a manifold (curve) which is overflowing invariant and which is $C^r \mathcal{O}(\epsilon)$ close to the *singular approximation*. It is easy to check that in the vicinity of the *singular approximation* which corresponds to the second term of the right-hand-side of Equation (9) we have:

$$L_{\vec{X}} \phi(\vec{X}) = \mathrm{Tr}[J] \phi(\vec{X}) \, .$$

Moreover, it can be shown that in the location of the points where the local *curvature* vanishes, i.e., where $\varphi(x, y) = 0$. Equation (9) can be written:

$$L_{\vec{X}} \phi(\vec{X}) = 0 \, .$$

So, according to Theorem 1 and 2, we can claim that the manifold defined by $\varphi(x, y) = 0$ is an *invariant curve* with respect to the flow of system (7) and is a *local first integral* of this system.

### 5.2 Lorenz model

The purpose of the model established by Edward Lorenz [6] was in the beginning to analyze the impredictible behaviour of weather. It most widespread form is as follows:

$$\vec{V} = \begin{pmatrix} \dfrac{\mathrm{d}x}{\mathrm{d}t} \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} \\ \dfrac{\mathrm{d}z}{\mathrm{d}t} \end{pmatrix} = \Im \begin{pmatrix} f(x, y, z) \\ g(x, y, z) \\ h(x, y, z) \end{pmatrix} = \begin{pmatrix} \sigma(y - x) \\ -xz + rx - y \\ xy - \beta z \end{pmatrix} \tag{10}$$

with $\sigma$, $r$ and $\beta$ are real parameters: $\sigma = 10$, $\beta = \frac{8}{3}$, $r = 28$ and where the functions $f$, $g$ and $h$ are infinitely differentiable with respect to all $x_i$, and $t$, i.e., are $C^\infty$ functions in a compact E included in $\mathbb{R}^3$ and with values in $\mathbb{R}$. According to Proposition 1, the location of the points where the local torsion vanishes leads to an equation which for place reasons can not be expressed. Let's name it as previously:

$$\varphi(x, y, z) \, . \tag{11}$$

According to Theorem 1 (Cf. Appendix for details), the Lie derivative of Equation (11) may be written:

$$L_{\vec{X}} \phi(\vec{X}) = \mathrm{Tr}[J] \phi(\vec{X}) + P(\vec{V} \cdot \vec{\gamma}) \tag{12}$$

where $P$ is a polynomial function of both vectors $\vec{V}$ and $\vec{\gamma}$. Let's plot the function $\phi(x, y, z)$ and its Lie derivative $L_{\vec{X}}\phi(\vec{X})$ and the *attractor* corresponding to system (10):

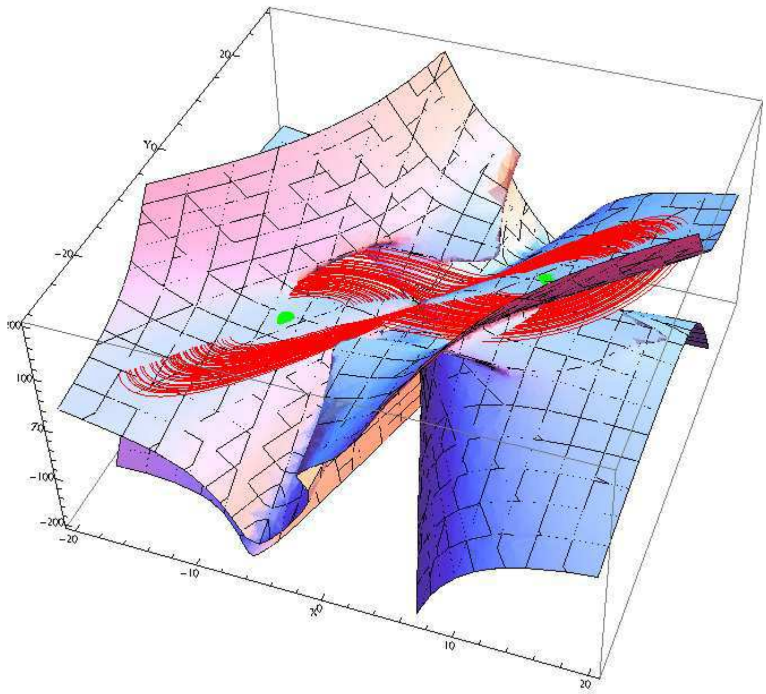

Figure 2: Lorenz model.

It is obvious that the function $\phi(x, y, z)$ defining a manifold (surface) is merged into the corresponding to its Lie derivative. It is easy to check that in the vicinity of the manifold $\phi(x, y, z)$ Equation (12) reduces to:

$$L_{\vec{X}}\phi(\vec{X}) = \mathrm{Tr}[J]\phi(\vec{X})\,.$$

Moreover, it can be shown that in the location of the points where the local torsion vanishes, i.e., where $\phi(x, y, z) = 0$ Equation (12) can be written:

$$L_{\vec{X}}\phi(\vec{X}) = 0$$

So, according to Theorem 1 and 2, we can claim that the manifold defined by $\phi(x, y, z) = 0$ is an invariant surface with respect to the flow of system (10) and is a local first integral of this system.

## 6 Discussion

In this work, existence of *invariant manifolds* which represent local first integrals of two (resp. three) dimensional dynamical systems defined by (1) has been established. From these two characteristics it can be stated that the former implies that such manifolds are representing the stable part of the trajectory curves in the phase space and from the latter that they are restoring the loss determinism inherent to the non-integrability feature of such systems. Moreover, while considering that dynamical systems defined by (1) include complex systems, it is possible to apply this method to various models of ecology (predator-prey models), neuroscience (neuronal bursting models), molecular biology (enzyme kinetics models)... Research of such invariant manifolds in coupled systems or in systems of higher dimension (four and more) would be of great interest.

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# Appendix

First of all, let's recall the following results:

$$L_{\vec{X}}\|\vec{u}\| = \frac{\mathrm{d}\|\vec{u}\|}{\mathrm{d}t} = \frac{\vec{u}\cdot\dot{\vec{u}}}{\|\vec{u}\|} \quad \text{(A-1)}$$

**Two-dimensional dynamical system**

Let's pose: $\varphi(\vec{X}) = \|\vec{\gamma}\wedge\vec{V}\|$. According to (A-1) the Lie derivative of this expression may be written:

$$L_{\vec{X}}\varphi(\vec{X}) = \frac{\mathrm{d}\|\vec{\gamma}\wedge\vec{V}\|}{\mathrm{d}t} = \frac{\left(\vec{\gamma}\wedge\vec{V}\right)\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{\gamma}\wedge\vec{V}\right)}{\|\vec{\gamma}\wedge\vec{V}\|} \quad \text{(A-2)}$$

where $\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{\gamma}\wedge\vec{V}\right) = \dot{\vec{\gamma}}\wedge\vec{V}$.

According to Equation (2) the Lie derivative of the acceleration vector may be written:

$$\dot{\vec{\gamma}} = J\vec{\gamma} + \frac{\mathrm{d}J}{\mathrm{d}t}\vec{V} \quad \text{(A-3)}$$

it leads to:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{\gamma}\wedge\vec{V}\right) = \dot{\vec{\gamma}}\wedge\vec{V} &= \left(J\vec{\gamma} + \frac{\mathrm{d}J}{\mathrm{d}t}\vec{V}\right)\wedge\vec{V} \\ &= J\vec{\gamma}\wedge\vec{V} + \frac{\mathrm{d}J}{\mathrm{d}t}\vec{V}\wedge\vec{V}\end{aligned} \quad \text{(A-4)}$$

Using the following identity:

$$(J\vec{a})\wedge\vec{b} + \vec{a}\wedge(J\vec{b}) = \mathrm{Tr}(J)(\vec{a}\wedge\vec{b})$$

it can be established that:

$$J\vec{\gamma}\wedge\vec{V} = \mathrm{Tr}(J)(\vec{\gamma}\wedge\vec{V})$$

So, expression (A-2) may be written:

$$L_{\vec{X}}\varphi(\vec{X}) = \frac{1}{\|\vec{\gamma}\wedge\vec{V}\|}\left(\mathrm{Tr}(J)\left(\vec{\gamma}\wedge\vec{V}\right)\cdot\left(\vec{\gamma}\wedge\vec{V}\right) + \left(\frac{\mathrm{d}J}{\mathrm{d}t}\vec{V}\wedge\vec{V}\right)\cdot\left(\vec{\gamma}\wedge\vec{V}\right)\right) \quad \text{(A-5)}$$

Let's note that: $\left(\vec{\gamma}\wedge\vec{V}\right)\cdot\left(\vec{\gamma}\wedge\vec{V}\right) = \|\vec{\gamma}\wedge\vec{V}\|^2$ and that: $\vec{\beta} = \dfrac{\vec{\gamma}\wedge\vec{V}}{\|\vec{\gamma}\wedge\vec{V}\|}$.

So, equation (A-5) leads to:

$$L_{\vec{X}}\varphi(\vec{X}) = \mathrm{Tr}(J)\|\vec{\gamma}\wedge\vec{V}\| + \left(\frac{\mathrm{d}J}{\mathrm{d}t}\vec{V}\wedge\vec{V}\right)\cdot\vec{\beta} \quad \text{(A-6)}$$

Since vector $\frac{\mathrm{d}J}{\mathrm{d}t}\vec{V}\wedge\vec{V}$ has a unique coordinate according to the $\vec{\beta}$-direction and since we have posed: $\varphi(\vec{X}) = \|\vec{\gamma}\wedge\vec{V}\|$, expression (A-6) may finally be written:

$$L_{\vec{X}}\varphi(\vec{X}) = \mathrm{Tr}(J)\varphi(\vec{X}) + \left\|\frac{\mathrm{d}J}{\mathrm{d}t}\vec{V}\wedge\vec{V}\right\| \quad \text{(A-7)}$$

**Three-dimensional dynamical system**

Let's pose: $\varphi(\vec{X}) = \dot{\vec{\gamma}}\cdot\left(\vec{\gamma}\wedge\vec{V}\right)$. The Lie derivative of this expression may be written:

$$L_{\vec{X}}\varphi(\vec{X}) = \frac{\mathrm{d}\left[\dot{\gamma}\cdot\left(\vec{\gamma}\wedge\vec{V}\right)\right]}{\mathrm{d}t} \quad \text{(A-8)}$$

According to $\dfrac{\mathrm{d}}{\mathrm{d}t}\left[\dot{\gamma}\cdot\left(\vec{\gamma}\wedge\vec{V}\right)\right] = \ddot{\vec{\gamma}}\cdot\left(\vec{\gamma}\wedge\vec{V}\right)$, it leads to:

$$L_{\vec{X}}\varphi(\vec{X}) = \frac{\mathrm{d}\left[\dot{\gamma}\cdot\left(\vec{\gamma}\wedge\vec{V}\right)\right]}{\mathrm{d}t} = \ddot{\vec{\gamma}}\cdot\left(\vec{\gamma}\wedge\vec{V}\right) \quad \text{(A-9)}$$

The Lie derivative of expression (A-3) leads to:

$$\ddot{\vec{\gamma}} = J\dot{\vec{\gamma}} + 2\frac{\mathrm{d}J}{\mathrm{d}t}\vec{\gamma} + \frac{\mathrm{d}^2J}{\mathrm{d}t^2}\vec{V}$$

Thus, expression (A-9) reads:

$$\begin{aligned} L_{\vec{X}}\varphi(\vec{X}) = \; & \left(J\dot{\vec{\gamma}}\right)\cdot\left(\vec{\gamma}\wedge\vec{V}\right) \\ & +\left(2\frac{\mathrm{d}J}{\mathrm{d}t}\vec{\gamma}+\frac{\mathrm{d}^2J}{\mathrm{d}t^2}\vec{V}\right)\cdot\left(\vec{\gamma}\wedge\vec{V}\right) \end{aligned} \tag{A-10}$$

It can also be established that:

$$\left(J^2\vec{\gamma}\right)\cdot\left(\vec{\gamma}\wedge\vec{V}\right)=\mathrm{Tr}(J)\left(J\vec{\gamma}\right)\cdot\left(\vec{\gamma}\wedge\vec{V}\right)$$

So, since we have posed: $\varphi(\vec{X})=\dot{\vec{\gamma}}\cdot\left(\vec{\gamma}\wedge\vec{V}\right)$, expression (A-10) may finally be written:

$$\begin{aligned} L_{\vec{X}}\varphi(\vec{X}) = \; & \mathrm{Tr}(J)\varphi(\vec{X})+\left(-\mathrm{Tr}(J)\frac{\mathrm{d}J}{\mathrm{d}t}\vec{V}\right. \\ & \left. J\frac{\mathrm{d}J}{\mathrm{d}t}\vec{V}+2\frac{\mathrm{d}J}{\mathrm{d}t}\vec{\gamma}+\frac{\mathrm{d}^2J}{\mathrm{d}t^2}\vec{V}\right)\cdot\left(\vec{\gamma}\wedge\vec{V}\right) \end{aligned} \tag{A-11}$$