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FLOW CURVATURE METHOD APPLIED TO CANARD EXPLOSION

JEAN-MARC GINOUX[1] AND JAUME LLIBRE[2]

Abstract. The aim of this work is to establish that the bifurcation parameter value leading to a *canard explosion* in dimension two obtained by the so-called *Geometric Singular Perturbation Method* can be found according to the *Flow Curvature Method*. This result will be then exemplified with the classical Van der Pol oscillator.

1. Introduction

The classical geometric theory of differential equations developed originally by Andronov [1], Tikhonov [30] and Levinson [23] stated that *singularly perturbed systems* possess *invariant manifolds* on which trajectories evolve slowly, and toward which nearby orbits contract exponentially in time (either forward or backward) in the normal directions. These manifolds have been called asymptotically stable (or unstable) *slow invariant manifolds*[1]. Then, Fenichel [11, 12, 13, 14] theory[2] for the *persistence of normally hyperbolic invariant manifolds* enabled to establish the *local invariance* of *slow invariant manifolds* that possess both expanding and contracting directions and which were labeled *slow invariant manifolds*.

During the last century, various methods have been developed to compute the *slow invariant manifold* or, at least an asymptotic expansion in power of ε.

The seminal works of Wasow [32], Cole [6], O'Malley [25, 26] and Fenichel [11, 12, 13, 14] to name but a few, gave rise to the so-called *Geometric Singular Perturbation Method*. According to this theory, existence as well as local invariance of the *slow invariant manifold* of *singularly perturbed systems* has been stated. Then, the determination of the *slow invariant manifold* equation turned into a regular perturbation problem in which one generally expected the asymptotic validity of such expansion to breakdown [26].

Recently a new approach of n-dimensional singularly perturbed dynamical systems of ordinary differential equations with two time scales, called *Flow Curvature Method* has been developed [17]. In dimension two and three, it consists in considering the *trajectory curves* integral of such systems as *plane* or *space* curves. Based on the use of local metrics properties of *curvature* (*first curvature*) and *torsion* (*second curvature*) resulting from the *Differential Geometry*, this method which does not require the use of asymptotic expansions, states that the location of the points

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[1]In other articles the *slow manifold* is the approximation of order $O(\varepsilon)$ of the *slow invariant manifold*.

[2]The theory of invariant manifolds for an ordinary differential equation is based on the work of Hirsch, *et al.* [19]

where the local *curvature* (resp. *torsion*) of *trajectory curves* of such systems, vanishes, directly provides an approximation of the *slow invariant manifold* associated with two-dimensional (resp. three-dimensional) *singularly perturbed systems* up to suitable order $O(\varepsilon^2)$ (resp. $O(\varepsilon^3)$). This method gives an implicit non intrinsic equation, because it depends on the euclidean metric.

Solutions of "canard" type have been discovered by a group of French mathematicians [2] in the beginning of the eighties while they were studying relaxation oscillations in the classical Van der Pol's equation (with a constant forcing term) [31]. They observed, within a small range of the control parameter, a fast transition for the amplitude of the limit cycle varying suddenly from small amplitude to a large amplitude. Due to the fact that the shape of the limit cycle in the phase plane looks as a duck they called it "canard cycle". Hence, they named this new phenomenon "canard explosion[3]" and triggered a "duck-hunting".

Many methods have been developed to analyze "canard" solution such as non-standard analysis [2, 8], matched asymptotic expansions [10], or the blow-up technique [9, 22, 28] which extends the *Geometric Singular Perturbation Method* [11, 12, 13, 14].

Meanwhile, two other geometric approaches have been proposed. The first, elaborated by [4] involves *inflection curves*, while the second makes use of the *curvature* of the trajectory curve, integral of any n-dimensional singularly perturbed dynamical system [16, 17]. This latter, entitled *Flow Curvature Method* will be used in this work in order to compute the bifurcation parameter value leading to a *canard explosion*. Moreover, the total correspondence between the results obtained in this paper for two-dimensional singularly perturbed dynamical systems such as Van der Pol oscillator and those previously established by [2] will enable to highlight another link between the *Flow Curvature Method* and the *Geometric Singular Perturbation Method*.

2. Singularly perturbed systems

According to Tikhonov [30], Takens [29], Jones [20] and Kaper [21] *singularly perturbed systems* may be defined such as:

$$
\begin{aligned}
\vec{x}' &= \vec{f}\left(\vec{x}, \vec{z}, \varepsilon\right), \\
\vec{z}' &= \varepsilon \vec{g}\left(\vec{x}, \vec{z}, \varepsilon\right).
\end{aligned}
\tag{1}
$$

where $\vec{x} \in \mathbb{R}^m$, $\vec{z} \in \mathbb{R}^p$, $\varepsilon \in \mathbb{R}^+$, and the prime denotes differentiation with respect to the independent variable t. The functions $\vec{f}$ and $\vec{g}$ are assumed to be C^∞ functions[4] of $\vec{x}$, $\vec{z}$ and ε in $U \times I$, where U is an open subset of $\mathbb{R}^m \times \mathbb{R}^p$ and I is an open interval containing $\varepsilon = 0$.

In the case when $0 < \varepsilon \ll 1$, i.e., is a small positive number, the variable $\vec{x}$ is called *fast* variable, and $\vec{z}$ is called *slow* variable. Using Landau's notation: $O\left(\varepsilon^k\right)$ represents a function f of x and ε such that $f(u, \varepsilon)/\varepsilon^k$ is bounded for positive ε going to zero, uniformly for u in the given domain.

[3]According to Krupa and Szmolyan [22, p. 312] this terminology has been introduced in chemical and biological literature by Br∅ns and Bar-Eli [3, p. 8707] to denote a sudden change of amplitude and period of oscillations under a very small range of control parameter.

[4]In certain applications these functions will be supposed to be C^r, $r \geqslant 1$.

It is used to consider that generally $\vec{x}$ evolves at an $O(1)$ rate; while $\vec{z}$ evolves at an $O(\varepsilon)$ *slow* rate. Reformulating system (1) in terms of the rescaled variable $\tau = \varepsilon t$, we obtain

$$\begin{aligned} \varepsilon\dot{\vec{x}} &= \vec{f}(\vec{x}, \vec{z}, \varepsilon), \\ \dot{\vec{z}} &= \vec{g}(\vec{x}, \vec{z}, \varepsilon). \end{aligned} \tag{2}$$

The dot represents the derivative with respect to the new independent variable τ.

The independent variables t and τ are referred to the *fast* and *slow* times, respectively, and (1) and (2) are called the *fast* and *slow* systems, respectively. These systems are equivalent whenever $\varepsilon \neq 0$, and they are labeled *singular perturbation problems* when $0 < \varepsilon \ll 1$. The label "singular" stems in part from the discontinuous limiting behavior in system (1) as $\varepsilon \to 0$.

In such case system (2) leads to a differential-algebraic system called *reduced slow system* whose dimension decreases from $m + p = n$ to p. Then, the *slow* variable $\vec{z} \in \mathbb{R}^p$ partially evolves in the submanifold M_0 called the *critical manifold*[5] and defined by

$$M_0 := \left\{ (\vec{x}, \vec{z}) : \vec{f}(\vec{x}, \vec{z}, 0) = \vec{0} \right\}. \tag{3}$$

When $D_x f$ is invertible, thanks to implicit function theorem, M_0 is given by the graph of a C^∞ function $\vec{x} = \vec{F}_0(\vec{z})$ for $\vec{z} \in D$, where $D \subseteq \mathbb{R}^p$ is a compact, simply connected domain and the boundary of D is an $(p-1)$–dimensional C^∞ submanifold[6].

According to Fenichel theory [11, 12, 13, 14] if $0 < \varepsilon \ll 1$ is sufficiently small, then there exists a function $\vec{F}(\vec{z}, \varepsilon)$ defined on D such that the manifold

$$M_\varepsilon := \left\{ (\vec{x}, \vec{z}) : \vec{x} = \vec{F}(\vec{z}, \varepsilon) \right\}, \tag{4}$$

is locally invariant under the flow of system (1). Moreover, there exist perturbed local stable (or attracting) M_a and unstable (or repelling) M_r branches of the *slow invariant manifold* M_ε. Thus, normal hyperbolicity of M_ε is lost via a saddle-node bifurcation of the *reduced slow system* (2).

Definition 1. *A "canard" is a solution of a singularly perturbed dynamical system following the attracting branch M_a of the slow invariant manifold, passing near a bifurcation point located on the fold of the critical manifold, and then following the repelling branch M_r of the slow invariant manifold during a considerable amount of time.*

Geometrically a *maximal canard* corresponds to the intersection of the attracting and repelling branches $M_a \cap M_r$ of the slow manifold in the vicinity of a non-hyperbolic point. Canards are a special class of solution of singularly perturbed dynamical systems for which normal hyperbolicity is lost.

[5]It corresponds to the approximation of the slow invariant manifold, with an error of $O(\varepsilon)$.

[6]The set D is overflowing invariant with respect to (2) when $\varepsilon = 0$.

3. Geometric Singular Perturbation Method

Earliest geometric approaches to *singularly perturbed dynamical systems* have been developed by Cole [6], O'Malley [25, 26], Fenichel [11, 12, 13, 14] for the determination of the *slow manifold* equation.

Geometric Singular Perturbation Method is based on the following assumptions and theorem stated by Nils Fenichel in the middle of the seventies[7].

3.1. **Assumptions.**

- (H_1) Functions $\vec{f}$ and $\vec{g}$ are C^∞ functions in $U \times I$, where U is an open subset of $\mathbb{R}^m \times \mathbb{R}^p$ and I is an open interval containing $\varepsilon = 0$.
- (H_2) There exists a set M_0 that is contained in $\{(\vec{x}, \vec{z}) : \vec{f}(\vec{x}, \vec{z}, 0) = 0\}$ such that M_0 is a compact manifold with boundary and M_0 is given by the graph of a C^1 function $\vec{x} = \vec{F}_0(\vec{z})$ for $\vec{z} \in D$, where $D \subseteq \mathbb{R}^p$ is a compact, simply connected domain and the boundary of D is an $(p-1)$–dimensional C^∞ submanifold. Finally, the set D is overflowing invariant with respect to (2) when $\varepsilon = 0$.
- (H_3) M_0 is normally hyperbolic relative to the *reduced fast system* and in particular it is required for all points $\vec{p} \in M_0$, that there are k (resp. l) eigenvalues of $D_{\vec{x}}\vec{f}(\vec{p}, 0)$ with positive (resp. negative) real parts bounded away from zero, where $k + l = m$.

Theorem 2 (Fenichel's Persistence Theorem)**.** *Let system* (1) *satisfying the conditions* $(H_1) - (H_3)$*. If* $\varepsilon > 0$ *is sufficiently small, then there exists a function* $\vec{F}(\vec{z}, \varepsilon)$ *defined on* D *such that the manifold* $M_\varepsilon = \{(\vec{x}, \vec{z}) : \vec{x} = \vec{F}(\vec{z}, \varepsilon)\}$ *is locally invariant under* (1)*. Moreover,* $\vec{F}(\vec{z}, \varepsilon)$ *is* C^r*, and* M_ε *is* C^r $O(\varepsilon)$ *close to* M_0*. In addition, there exist perturbed local stable and unstable manifolds of* M_ε*. They are unions of invariant families of stable and unstable fibers of dimensions* l *and* k*, respectively, and they are* C^r $O(\varepsilon)$ *close to their counterparts.*

Proof. See [11], [20] and [21]. □

3.2. **Invariance.**

Generally, Fenichel theory enables to turn the problem for explicitly finding functions $\vec{x} = \vec{F}(\vec{z}, \varepsilon)$ whose graphs are locally *slow invariant manifolds* M_ε of system (1) into regular perturbation problem. Invariance of the manifold M_ε implies that $\vec{F}(\vec{z}, \varepsilon)$ satisfies:

$$\varepsilon D_{\vec{z}}\vec{F}(\vec{z}, \varepsilon)\, \vec{g}\left(\vec{F}(\vec{z}, \varepsilon), \vec{z}, \varepsilon\right) = \vec{f}\left(\vec{F}(\vec{z}, \varepsilon), \vec{z}, \varepsilon\right). \tag{5}$$

Then, plugging the perturbation expansion:

$$\vec{F}(\vec{z}, \varepsilon) = \sum_{i=0}^{N-1} \vec{F}_i(\vec{z})\, \varepsilon^i + O\left(\varepsilon^N\right)$$

into (5) enables to solve order by order for $\vec{F}(\vec{z}, \varepsilon)$.

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[7] For an introduction to Geometric Singular Perturbation Method see [21].

Taylor series expansion for $\vec{f}\left(\vec{F}\left(\vec{z},\varepsilon\right),\vec{z},\varepsilon\right)$ up to terms of order two in ε leads at order ε^0 to

$$\vec{f}\left(\vec{F}_0\left(\vec{z}\right),\vec{z},0\right)=\vec{0} \tag{6}$$

which defines $\vec{F}_0\left(\vec{z}\right)$ due to the invertibility of $D_{\vec{x}}\vec{f}$ and the *Implicit Function Theorem*.

At order ε^1 we have:

$$D_{\vec{z}}\vec{F}_0\vec{g}\left(\vec{F}_0,\vec{z},0\right)=D_{\vec{x}}\vec{f}\left(\vec{F}_0,\vec{z},0\right)\vec{F}_1+\frac{\partial\vec{f}}{\partial\varepsilon}\left(\vec{F}_0,\vec{z},0\right), \tag{7}$$

which yields $\vec{F}_1\left(\vec{z}\right)$ and so forth.

$$D_{\vec{x}}\vec{f}\left(\vec{F}_0,\vec{z},0\right)\vec{F}_1=D_{\vec{z}}\vec{F}_0\vec{g}\left(\vec{F}_0,\vec{z},0\right)-\frac{\partial\vec{f}}{\partial\varepsilon}\left(\vec{F}_0,\vec{z},0\right). \tag{8}$$

So, regular perturbation theory enables to build locally *slow invariant manifolds* M_ε. But for high-dimensional *singularly perturbed systems slow invariant manifold* asymptotic equation determination leads to tedious calculations.

Proof. For application of this technique see [14]. □

3.3. **Slow invariant manifold and canards.** A manifold of canards is an invariant manifold, where first approximation is M_0. For two-dimensional singularly perturbed dynamical systems with just one fast variable (x) and one slow variable (y), canards are non generic according to Krupa and Szmolyan [22] and *maximal canards* can only occur in such systems only for discrete values of a control parameter μ. It means that in dimension two a one parameter family of singularly perturbed systems is needed to exhibit canard phenomenon. Because along a canard, the differential $D_x f$ is not always invertible, we can not write the manifold of canards as $x=F(y,\varepsilon)$. Thus, we will suppose that $D_y f$ is invertible and we will try to compute the canard as $y=F(x,\mu,\varepsilon)$. See [5] for a theory of this identification of formal series. We consider the following *singularly perturbed dynamical system*:

$$\begin{aligned}\varepsilon\dot{x}&=f(x,y,\mu,\varepsilon),\\ \dot{y}&=g(x,y,\mu,\varepsilon),\end{aligned}$$

with $x,y\in\mathbb{R}$, i.e. $(m,p)=(1,1)$ and we suppose that due to the nature of the problem perturbation expansions of the canard and of the canard value read:

$$y=F(x,\varepsilon)=\sum_{i=0}^{N-1}F_i(x)\varepsilon^i+O\left(\varepsilon^N\right)\quad\text{and}\quad\mu(\varepsilon)=\sum_{i=0}^{N-1}\mu_i\varepsilon^i+O\left(\varepsilon^N\right).$$

According to Eq. (5) invariance of the manifold M_ε reads:

$$\left(\frac{\partial F}{\partial x}(x,\varepsilon)\right)f\left(x,F(x,\varepsilon),\mu(\varepsilon),\varepsilon\right)=\varepsilon g\left(x,F(x,\varepsilon),\mu(\varepsilon),\varepsilon\right). \tag{9}$$

To avoid technical complications in the computations below, we assume that, at order $O(\varepsilon^0)$, the critical manifold does not depend on the parameter μ.

Indeed,

$$\frac{\partial f}{\partial \mu}(x, F_0(x), \mu_0, 0) = 0$$

Then, solving equation (9) order by order provides at:

Order ε^0

$$\frac{\partial F_0}{\partial x}(x) f(x, F_0(x), \mu_0, 0) = 0 \quad \Leftrightarrow \quad f(x, F_0(x), \mu_0, 0) = 0. \tag{10}$$

because the function $\dfrac{\partial F_0}{\partial x}(x)$ is almost everywhere non zero. Indeed, the function F_0 is given by the implicit function theorem. In what follows f, g, and their derivatives are evaluated at $(x, F_0(x), \mu_0, 0)$, and F_0, F_1 and F_2 are evaluated at x.

Order ε^1

$$F_0' \left(\frac{\partial f}{\partial y} F_1 + \frac{\partial f}{\partial \mu} \mu_1 + \frac{\partial f}{\partial \varepsilon} \right) + F_1' f = g.$$

Since according to what has been stated before, we have:

$$F_1 = \frac{\dfrac{g}{F_0'} - \dfrac{\partial f}{\partial \varepsilon}}{\dfrac{\partial f}{\partial y}} \tag{11}$$

A priori, this function is singular at the bifurcation point x_0 of the fast system, because F_0' vanishes at this point. To avoid this singularity in function F_1, the relation $g(x_0, F(x_0), \mu_0, 0) = 0$ is needed. Whith an appropriate hypothesis on $\dfrac{\partial g}{\partial \mu}$, it gives a value for μ_0.

Higher order The computation can be done with the same arguments. When condition of order k are studied, we have to fix F_k, and to avoid singularity in F_k we have to fix μ_{k-1}. An example will be done in the next paragraph.

3.4. **Van der Pol's "canards".** Van der Pol system

$$\begin{aligned} \varepsilon \dot{x} &= f(x, y) = x + y - \frac{x^3}{3}, \\ \dot{y} &= g(x, y) = \mu - x, \end{aligned} \tag{12}$$

satisfies Fenichel's assumptions (H_1)–(H_3) except on the points $(x, y) = \pm(1, -\frac{2}{3})$. The critical manifold is the cubic $y = x^3/3 - x$. Thus, the problem is to find a function $y = F(x, \varepsilon)$ whose graph is locally the *slow invariant manifold* M_ε of the Van der Pol system. We write:

$$F(x, \varepsilon) = F_0(x) + \varepsilon F_1(x) + \varepsilon^2 F_2(x) + O\left(\varepsilon^3\right)$$

and

$$\mu = \mu_0 + \varepsilon \mu_1 + \varepsilon^2 \mu_2 + O\left(\varepsilon^3\right).$$

The identification we have to perform is

$$\sum_{i=0} \frac{\partial F_i}{\partial x}\varepsilon^i \left(\sum_{i=0} F_i \varepsilon^i - \left(\frac{x^3}{3} - x \right) \right) = \sum_{i=0} \mu_i \varepsilon^{i+1} - \varepsilon x$$

Then, solving order by order provides at:

Order ε^0

$$F_0(x) = \frac{x^3}{3} - x.$$

Order ε^1

$$F_1(x) = -\frac{x - \mu_0}{x^2 - 1} \tag{13}$$

This function is singular at the fold point $x_0 = 1$ corresponding to the Hopf bifurcation point of the fast system[8]. So, to avoid this singularity in function $F_1(x)$ we pose: $\mu_0 = 1$ and thus we have: $F_1(x) = \dfrac{-1}{1+x}$.

Order ε^2

$$F_2(x) = \frac{\mu_1 + \dfrac{(x-\mu_0)(x^2+1-2\mu_0 x)}{(x^2-1)^3}}{x^2 - 1} \tag{14}$$

Taking into account that $\mu_0 = 1$ and, in order to avoid singularity in $F_2(x)$, we find that $\mu_1 = -\dfrac{1}{8}$ and so $F_2(x) = -\dfrac{x^2+4x+7}{8(1+x)^4}$.

Order ε^3

Using the same process, a tedious computation (or, better a computation with the help of a computer) leads to $\mu_2 = -\dfrac{3}{32}$, $\mu_3 = -\dfrac{173}{1024}$. Thus, the bifurcation parameter value leading to canard solutions reads:

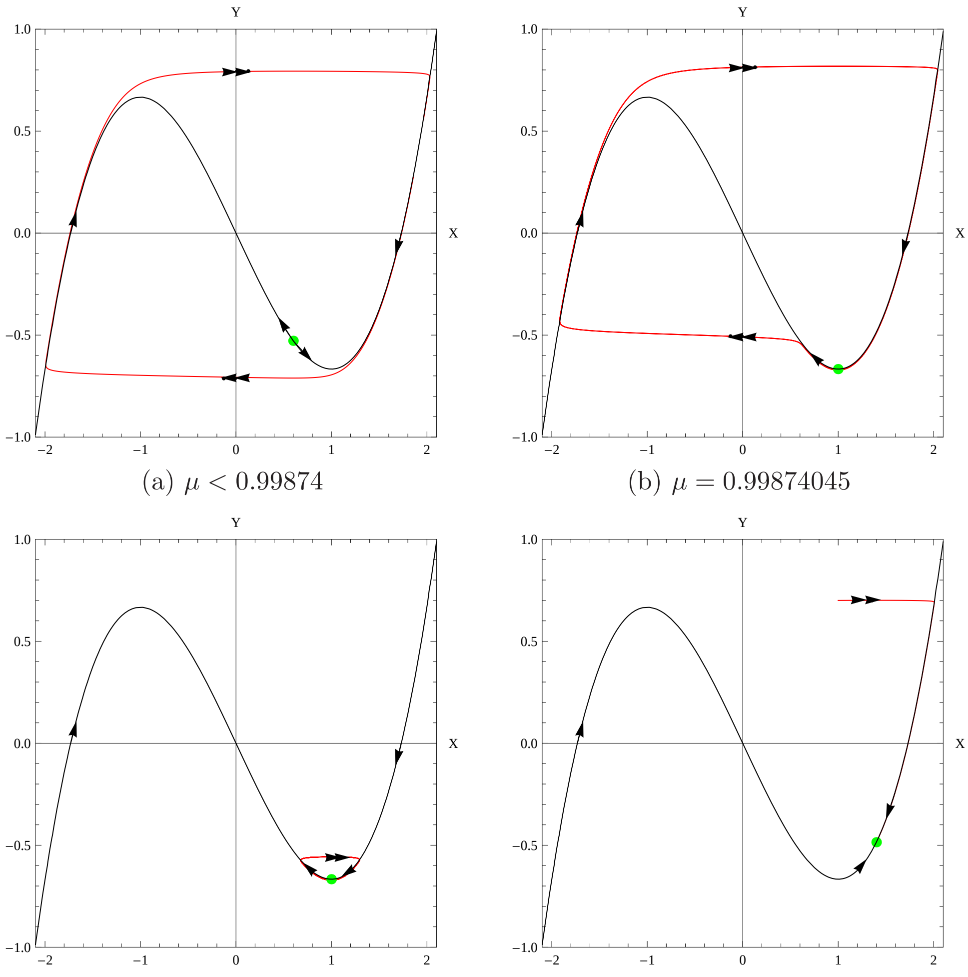
$$\mu = 1 - \frac{1}{8}\varepsilon - \frac{3}{32}\varepsilon^2 + O\left(\varepsilon^3\right)$$

Then, for $\varepsilon = 0.01$ one finds again the value obtained numerically by Benoît *et al.* [2, p.99] and Diener [8]:

$$\mu = 0.99874$$

[8]Due to the symmetry of the vector field: $(-x, -y, -\mu) \to (x, y, \mu)$ the same computation could have been done on the fold point $x_0 = -1$ in the vicinity of which a "canard explosion" also takes place.

The phenomenon of "canard explosion" of Van der Pol system (12) with $\varepsilon = 0.01$ is exemplified on Fig. 1. where the periodic solution has been plotted in red, the critical manifold in black and the positive fixed point in green. Double arrows indicate the *fast* motion while simple arrows indicate the *slow* motion. Exponentially small variations of the parameter value $\mu = 0.99874$ enable to exhibit the transition from relaxation oscillation (a) to small amplitude limit cycles (b) via canard cycles (c). Then, at the parameter value $\mu = 1$ corresponding to the Hopf bifurcation, the canard disappears (d).



(a) $\mu < 0.99874$ (b) $\mu = 0.99874045$

(c) $\mu = 0.998740451$ (d) $\mu > 1$

FIGURE 1. Transition from relaxation oscillation to canard explosion.

4. Flow Curvature Method

Recently, a new approach called *Flow Curvature Method* and based on the use of *Differential Geometry* properties of *curvatures* has been developed, see [16, 17]. According to this method, the highest *curvature of the flow*, i.e. the $(n-1)^{th}$ *curvature* of *trajectory curve* integral of n–dimensional dynamical system defines a *manifold* associated with this system and called *flow curvature manifold*. In the case of n–dimensional singularly perturbed dynamical system (1) for which with $\vec{x} \in \mathbb{R}^1$, $\vec{z} \in \mathbb{R}^{n-1}$, i.e. $(m,p) = (1, n-1)$ we have the following result.

Proposition 3. *The location of the points where the* $(n-1)^{th}$ *curvature of the flow, i.e. the curvature of the trajectory curve* $\vec{X}$*, integral of any* n*–dimensional singularly perturbed dynamical system vanishes, represents its* $(n-1)$*–dimensional slow manifold* M_ε *the equation of which reads*

$$\phi(\vec{X}, \varepsilon) = \dot{\vec{X}} \cdot (\ddot{\vec{X}} \wedge \dddot{\vec{X}} \wedge \ldots \wedge \overset{(n)}{\vec{X}}) = \det(\dot{\vec{X}}, \ddot{\vec{X}}, \dddot{\vec{X}}, \ldots, \overset{(n)}{\vec{X}}) = 0 \tag{15}$$

where $\overset{(n)}{\vec{X}}$ *represents the time derivatives up to order* n *of* $\vec{X} = (\vec{x}, \vec{z})^t$.

Proof. For proof of this proposition see [17, p. 185 and next] and below. □

Remark 4. *First, let's notice that with the Flow Curvature Method the slow manifold is defined by an implicit equation. Secondly, in the most general case of* n*–dimensional singularly perturbed dynamical system (1) for which* $\vec{x} \in \mathbb{R}^m$, $\vec{z} \in \mathbb{R}^p$ *the Proposition 3 still holds. In dimension three, the example of a Neuronal Bursting Model (NBM) for which* $(m,p) = (2,1)$ *has already been studied by Ginoux et al.* [15]*. In this particular case, one of the hypotheses of the Tihonov's theorem is not checked since the fast dynamics of the singular approximation has a periodic solution. Nevertheless, it has been established by Ginoux et al.* [15] *that the slow manifold can all the same be obtained while using the Flow Curvature Method. According to this method, the slow invariant manifold of a three-dimensional singularly perturbed dynamical system for which* $(m,p) = (1,2)$ *is given by the* 2^{nd} *curvature of the flow, i.e. the torsion. In the case of a Neuronal Bursting Model for which* $(m,p) = (2,1)$ *it has been stated by Ginoux et al.* [15] *that the slow manifold is then given by the* 1^{st} *curvature of the flow, i.e. the curvature. In such a case, the flow curvature manifold is defined by the location of the points where the three-dimensional pseudovector* $\dot{\vec{X}} \wedge \ddot{\vec{X}}$ *vanishes. This condition leads to a nonlinear system of three equations two of which being linearly independent. These two equations define a curve corresponding to the slow invariant manifold*[9]*. Thus, one can deduce that for a three-dimensional singularly perturbed dynamical system for which* (m,p) *the slow manifold is given by the* p^{th} *curvature of the flow.*

4.1. Invariance.

According to Schlomiuk [27] and Llibre *et al.* [24] the concept of *invariant manifold* has been originally introduced by Gaston Darboux [7, p. 71] in a memoir entitled: *Sur les équations différentielles algébriques du premier ordre et du premier degré* and can be stated as follows.

[9]See also Gilmore *et al.* [18]

Proposition 5. *The manifold defined by $\phi(\vec{X},\varepsilon)=0$ where ϕ is a C^1 in an open set U, is invariant with respect to the flow of* (1) *if there exists a C^1 function denoted by $\kappa(\vec{X},\varepsilon)$ and called cofactor which satisfies*

$$L_{\vec{V}}\phi(\vec{X},\varepsilon)=\kappa(\vec{X},\varepsilon)\phi(\vec{X},\varepsilon), \tag{16}$$

for all $\vec{X}\in U$, and with the Lie derivative operator defined as

$$L_{\vec{V}}\phi=\overrightarrow{V}\cdot\overrightarrow{\nabla}\phi=\sum_{i=1}^{n}\frac{\partial\phi}{\partial x_i}\dot{x}_i=\frac{d\phi}{dt}.$$

Proof. According to *Fenichel's Persistence Theorem* (see Th. 2) the *slow invariant manifold* M_ε may be written as an *explicit* function $\vec{x}=\vec{F}(\vec{z},\varepsilon)$, the invariance of which implies that $\vec{F}(\vec{z},\varepsilon)$ satisfies

$$\varepsilon D_{\vec{z}}\vec{F}(\vec{z},\varepsilon)\,\vec{g}\left(\vec{F}(\vec{z},\varepsilon),\vec{z},\varepsilon\right)=\vec{f}\left(\vec{F}(\vec{z},\varepsilon),\vec{z},\varepsilon\right) \tag{17}$$

We write the *slow manifold* M_ε as an *implicit function* by posing

$$\phi(\vec{x},\vec{z},\varepsilon)=\vec{x}-\vec{F}(\vec{z},\varepsilon)=\phi(\vec{X},\varepsilon). \tag{18}$$

According to *Darboux invariance theorem* M_ε is invariant if its Lie derivative reads

$$L_{\vec{V}}\phi(\vec{X},\varepsilon)=\kappa(\vec{X},\varepsilon)\phi(\vec{X},\varepsilon). \tag{19}$$

Plugging Eq. (18) into the Lie derivative (19) leads to

$$L_{\vec{V}}\phi(\vec{X},\varepsilon)=\dot{\vec{x}}-D_{\vec{z}}\vec{F}(\vec{z},\varepsilon)\dot{\vec{z}}=\kappa(\vec{X},\varepsilon)\phi(\vec{X},\varepsilon),$$

which may be written according to Eq. (2) as

$$L_{\vec{V}}\phi(\vec{X},\varepsilon)=\frac{1}{\varepsilon}(\vec{f}(\vec{X},\varepsilon)-\varepsilon D_{\vec{z}}\vec{F}(\vec{z},\varepsilon)\vec{g}(\vec{X},\varepsilon))=\kappa(\vec{X},\varepsilon)\phi(\vec{X},\varepsilon).$$

Evaluating this Lie derivative in the location of the points where $\phi(\vec{X},\varepsilon)=0$, i.e. $\vec{x}=\vec{F}(\vec{z},\varepsilon)$ leads to

$$L_{\vec{V}}\phi(\vec{F}(\vec{z},\varepsilon),\vec{z},\varepsilon)=\frac{1}{\varepsilon}\left(\vec{f}(\vec{F}(\vec{z},\varepsilon),\vec{z},\varepsilon)-\varepsilon D_{\vec{z}}\vec{F}(\vec{z},\varepsilon)\,\vec{g}(\vec{F}(\vec{z},\varepsilon),\vec{z},\varepsilon)\right)=0,$$

which is exactly identical to Eq. (18) used by Fenichel. □

Remark 6. *This last equation for the invariance of the manifold M_ε may be written in a simpler way which implies that $\phi(\vec{x},\vec{z},\varepsilon)$ satisfies*

$$\frac{d}{dt}\left[\phi(\vec{x},\vec{z},\varepsilon)\right]=0, \tag{20}$$

on the solutions of the differential system.

4.2. **Slow invariant manifold.** We consider again the following two-dimensional *singularly perturbed dynamical system*

$$\varepsilon \dot{x} = f(x,y),$$
$$\dot{y} = g(x,y),$$

and we suppose that due to the nature of the problem perturbation expansion reads

$$y = F(x,\varepsilon) = F_0(x) + \varepsilon F_1(x) + \varepsilon^2 F_2(x) + O\left(\varepsilon^3\right).$$

According to the *Flow Curvature Method* each function $\vec{F_i}(\vec{z})$ of this perturbation expansion may be found again starting from the *slow manifold implicit equation* (16) as stated in the next result.

Proposition 7. *The functions $F_i(x)$ of the slow invariant manifold associated with a two-dimensional singularly perturbed dynamical system are given by the following expressions*

$$F'_n(x) = \lim_{\varepsilon \to 0} \left[\frac{1}{n!} \frac{\partial^n a_{10}}{\partial \varepsilon^n} \right] \quad \textit{with } n \geq 0,$$
$$F_n(x) = \lim_{\varepsilon \to 0} \left[\frac{1}{n!} \frac{\partial^{n-1} a_{01}}{\partial \varepsilon^{n-1}} \right] \quad \textit{with } n \geq 1, \tag{21}$$

where

$$a_{10} = - \left. \frac{\dfrac{\partial \phi_i}{\partial x}}{\dfrac{\partial \phi_i}{\partial y}} \right|_{y=F(x,\varepsilon)} \quad \textit{and} \quad a_{01} = - \left. \frac{\dfrac{\partial \phi_i}{\partial \varepsilon}}{\dfrac{\partial \phi_i}{\partial y}} \right|_{y=F(x,\varepsilon)},$$

and

$$\phi_i(x,y,\varepsilon) = \frac{d^{i-1}}{dt^{i-1}} [\phi_1(x,y,\varepsilon)] \quad \textit{with} \quad i = 1,\ 2,\ \ldots,\ n,$$

where $\phi_i(x,y,\varepsilon)$ corresponds to the i^{th} order approximation in ε.

Proof. We have that

$$\phi(\vec{X},\varepsilon) = \left\| \dot{\vec{X}} \wedge \ddot{\vec{X}} \right\| = \det(\dot{\vec{X}}, \ddot{\vec{X}}) = 0 \quad \Leftrightarrow \quad \phi\left(x,y,\varepsilon\right) = 0.$$

Since, for a two-dimensional singularly perturbed dynamical systems this *slow manifold equation* is defined by the *second order tensor of curvature*, i.e. by a determinant involving the first and second time derivatives of the vector field $\vec{X}$, it corresponds to the first order approximation in ε of the *slow manifold* obtained with the *Geometric Singular Perturbation Method*. So, we denote it by

$$\phi_1(\vec{X},\varepsilon) = \left\| \dot{\vec{X}} \wedge \ddot{\vec{X}} \right\| = \det(\dot{\vec{X}}, \ddot{\vec{X}}) = 0.$$

A *third order tensor of curvature* can be easily given by the time derivative of $\phi_1(\vec{X},\varepsilon)$. We denote it by

$$\phi_2(\vec{X},\varepsilon) = \dot{\phi}_1(\vec{X},\varepsilon) = \det(\dot{\vec{X}}, \dddot{\vec{X}}) = 0.$$

Thus, $\phi_2(\vec{X},\varepsilon)$ corresponds to the second order approximation in ε. Using the same process, we consider the *slow manifold* $\phi_i(\vec{X},\varepsilon)$ which corresponds to the i^{th} order approximation in ε.

Writing the *total differential* of the *slow manifold* we obtain

$$d\phi_i\left(x,y,\varepsilon\right)=\frac{\partial\phi_i}{\partial x}dx+\frac{\partial\phi_i}{\partial y}dy+\frac{\partial\phi_i}{\partial\varepsilon}d\varepsilon=0. \tag{22}$$

Replacing in Eq. (23) dy by its *total differential* $dy=\dfrac{\partial F}{\partial x}dx+\dfrac{\partial F}{\partial\varepsilon}d\varepsilon$ yields

$$d\phi_i(x,y,\varepsilon)=\left(\frac{\partial\phi_i}{\partial x}+\frac{\partial\phi_i}{\partial y}\frac{\partial F}{\partial x}\right)dx+\left(\frac{\partial\phi_i}{\partial\varepsilon}+\frac{\partial\phi_i}{\partial y}\frac{\partial F}{\partial\varepsilon}\right)d\varepsilon. \tag{23}$$

According to Eq. (21) $\phi_i(x,y,\varepsilon)$ is *invariant* if and only if $d\phi_i(x,y,\varepsilon)=0$, i.e. if

$$\begin{aligned}\frac{\partial\phi_i}{\partial x}+\frac{\partial\phi_i}{\partial y}\frac{\partial F}{\partial x}=0\quad&\Leftrightarrow\quad\frac{\partial F}{\partial x}=-\frac{\dfrac{\partial\phi_i}{\partial x}}{\dfrac{\partial\phi_i}{\partial y}},\\\frac{\partial\phi_i}{\partial\varepsilon}+\frac{\partial\phi_i}{\partial y}\frac{\partial F}{\partial\varepsilon}=0\quad&\Leftrightarrow\quad\frac{\partial F}{\partial\varepsilon}=-\frac{\dfrac{\partial\phi_i}{\partial\varepsilon}}{\dfrac{\partial\phi_i}{\partial y}}.\end{aligned} \tag{24}$$

By replacing $y=F\left(x,\varepsilon\right)$ by its expression in both parts of Eq. (25) and by setting

$$a_{10}=-\left.\frac{\dfrac{\partial\phi_i}{\partial x}}{\dfrac{\partial\phi_i}{\partial y}}\right|_{y=F(x,\varepsilon)}\qquad\text{and}\qquad a_{01}=-\left.\frac{\dfrac{\partial\phi_i}{\partial\varepsilon}}{\dfrac{\partial\phi_i}{\partial y}}\right|_{y=F(x,\varepsilon)},$$

we have that

$$\begin{aligned}\frac{\partial F\left(x,\varepsilon\right)}{\partial x}&=F_0'(x)+\ \varepsilon F_1'(x)+O\left(\varepsilon^2\right)=a_{10},\\\frac{\partial F\left(x,\varepsilon\right)}{\partial\varepsilon}&=F_1(x)+2\varepsilon F_2(\varepsilon)+O\left(\varepsilon^2\right)=a_{01}.\end{aligned} \tag{25}$$

By using a *recurrence reasoning* it may be easily stated that the functions $F_i(x)$ of the *slow invariant manifold* associated with a two-dimensional singularly perturbed dynamical system are given by the expressions (21). $\square$

4.3. **Van der Pol's "canards".** We consider again the Van der Pol system (12). All functions $F_i(x)$ of the perturbation expansion may be deduced from the *slow manifold* equation defined by (16). But, since the determination of $F_3(x)$, i.e. the computation μ_2 requires a *third order tensor of curvature* we consider $\phi_3(\vec{X},\varepsilon)$ the second time derivative of $\phi_1(\vec{X},\varepsilon)$ which corresponds to the third order approximation in ε.

Thus, we find at

Order ε^0

$$F_0'(x) = \lim_{\varepsilon \to 0} [a_{10}] = -1 + x^2,$$

from which one deduces that

$$F_0(x) = \frac{x^3}{3} - x + C_0,$$

where the constant C_0 may be chosen in such a way that the *critical manifold* can be found again ($C_0 = 0$).

Order ε^1

$$F_1(x) = \lim_{\varepsilon \to 0} [a_{01}] = \frac{\mu_0 - x}{x^2 - 1}. \tag{26}$$

Thus, one find again exactly the same functions $F_1(x)$ as those given by *Geometric Singular Perturbation Method* (14) and of course the same value of the bifurcation parameter $\mu_0 = 1$.

Order ε^2

$$F_2(x) = \frac{\mu_1 + \dfrac{(x - \mu_0)(x^2 + 1 - 2\mu_0 x)}{(x^2 - 1)^3}}{x^2 - 1}. \tag{27}$$

Taking into account that $\mu_0 = 1$ we find again exactly the same functions $F_2(x)$ as those given by *Geometric Singular Perturbation Method* (15) and of course the same value of the bifurcation parameter $\mu_1 = -1/8$.

Order ε^3

A simple and direct computation leads to $\mu_2 = -\dfrac{3}{32}$. Thus, the bifurcation parameter value leading to canard solutions reads

$$\mu = 1 - \frac{1}{8}\varepsilon - \frac{3}{32}\varepsilon^2 + O\left(\varepsilon^3\right).$$

Then, for $\varepsilon = 0.01$ one finds again the value obtained by Benoît *et al.* [2, p.99] and Diener [8]

$$\mu = 0.99874...$$

A program made with Mathematica and available at: http://ginoux.univ-tln.fr enables to compute all order of approximations in ε of any two-dimensional singularly perturbed systems.

5. Conclusion

Thus, the bifurcation parameter value leading to a *canard explosion* in dimension two obtained by the so-called *Geometric Singular Perturbation Method* has been found again with the *Flow Curvature Method*. This result could be also extended to three-dimensional singularly perturbed dynamical systems such as the 3D-autocatalator in which canard phenomenon occurs.

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