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# KERDOCK-LIKE BENT FUNCTIONS 

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#### Abstract

We introduce bent functions similar to bent functions whose binary representative vectors are members of the famous Kerdock code.


## 1. Introduction

### 1.1. Elementary definitions.

$\mathbb{F}_{2}$ is the finite field of order 2 .
A m-boolean function is a map from $\mathbb{F}_{2}^{m}$ into $\mathbb{F}_{2}$.
Weight: $w(F)=\sharp\left\{v \in \mathbb{F}_{2}^{m} \mid F(v)=1\right\}$.
Derivative: $e \in \mathbb{F}_{2}^{m}\left(D_{e} F\right)(X)=F(X)+F(X+e)$.
Fourier coefficients:
$\hat{F}(v)=\sum_{X \in \mathbb{F}_{2}^{m}}(-1)^{F(X)+<v, X>}$ where $<,>$ inner product of $\mathbb{F}_{2}^{m}$.
The set $\left\{\hat{F}(v) \mid v \in \mathbb{F}_{2^{m}}\right\}$ is independant of the choice of $<,>$.
Definitions:
$F$ is bent if: $\forall v \in \mathbb{F}_{2}^{m}: \hat{F}(v)$ is in $\left\{-2^{m / 2}, 2^{m / 2}\right\}$.
Exist only when $m$ is even.
$F$ is near-benf if: $\forall v \in \mathbb{F}_{2^{m}}: \hat{F}(v)$ is in $\left\{-2^{(m+1) / 2}, 0,2^{(m+1) / 2}\right\}$.
Exist only when $m$ is odd.
Bent functions were introduced by Rothaus in [6]. They are interesting for Coding Theory, Cryptology and Sequences and were the topic of a lot of works. See for instance [2], [5] Chap. 14, [7], [1].
For further use we need the following Proposition.
Proposition 1. The distribution of the Fourier coefficients of $a$ (2t-1)-near bent function $f$ is:

$$
\begin{array}{ll}
\hat{f}(v)=2^{t} & \text { number of } v: 2^{2 t-3}+(-1)^{f(0)} 2^{t-2} \\
\hat{f}(v)=0 & \text { number of } v: 2^{2 t-2} \\
\hat{f}(v)=-2^{t} & \text { number of } v: 2^{2 t-3}-(-1)^{f(0)} 2^{t-2} .
\end{array}
$$

Proof. See Proposition 4 in [1]).

### 1.2. Special representations of boolean functions.

1) Using finite fields:
$\mathbb{F}_{2}^{m}$ identified with the field $\mathbb{F}_{2^{m}}$.

In this case the inner product $<,>$ of $\mathbb{F}_{2^{m}}$ is defined by:
$<a, x\rangle=\operatorname{tr}(a x)$ where $\operatorname{tr}$ is the trace of $\mathbb{F}_{2^{m}}$ over $\mathbb{F}_{2}$.
2) Representative vector (truth table)

Indexing $\mathbb{F}_{2^{m}}$ with any order $e_{0}, e_{1}, \ldots e_{2^{m}-1}$ the
representative vector of a m-boolean function $F$ is the binary vector $\left(F\left(e_{i}\right)_{i=0}^{2^{m}-1}\right)$.
This vector depends on the choice of the order of $\mathbb{F}_{2^{m}}$.
3) A two-variable representation.

This is the representation chosen by Kerdock to introduce his famous code.
We identify $\mathbb{F}_{2^{2 t}}$ with the product:
$\mathbb{F}_{2^{2 t-1}} \times \mathbb{F}_{2}=\left\{X=(u, \nu) \mid u \in \mathbb{F}_{2^{2 t-1}}, \nu \in \mathbb{F}_{2}\right\}$.
If $F$ is a $(2 t)$-boolean function then define two ( $2 t-1$ )-boolean functions $f_{0}, f_{1}$, such that $f_{0}(u)=F(u, 0)$ and $f_{1}(u)=F(u, 1)$.

The two-variable representation (TVR) of $F$ is defined by the 2 -variable polynomial: $\quad \phi_{F}(x, y)=(y+1) f_{0}(x)+y f_{1}(x)$

This is a representation of $F$ in the following sense. Since:
$\phi_{F}(u, 0)=f_{0}(u)=F(u, 0), \phi_{F}(u, 1)=f_{1}(u)=F(u, 1)$.
then if $X=(u, \nu)$, with $u=0$ or $u=1: F(X)=\phi_{F}(u, \nu)$.
Notation: $F=\left[f_{0}, f_{1}\right]$
Let $\alpha$ be a primitive root of $\mathbb{F}_{2^{k-1}}$. As order of $\mathbb{F}_{2^{2 t}}$ we choose:
$\mathbb{F}_{2^{2 t}}:(0,0),\left(\alpha^{0}, 0\right) \ldots\left(\alpha^{i}, 0\right) \ldots\left(\alpha^{n / 2-2}, 0\right) \mid(0,1),\left(\alpha^{0}, 1\right) \ldots\left(\alpha^{i}, 1\right) \ldots\left(\alpha^{n / 2-2}, 1\right)$ The representative vector of $F=\left[f_{0}, f_{1}\right]$ is:
$\left(f_{0}(0) \ldots \ldots f_{0}\left(\alpha^{i}\right) \ldots \ldots f_{0}\left(\alpha^{n / 2-2}\right) \quad f_{1}(0) \ldots \ldots f_{1}\left(\alpha^{i}\right) \ldots \ldots f_{1}\left(\alpha^{n / 2-2}\right)\right)$

### 1.3. From Near-bent to Bent.

We now characterize the $f_{0}, f_{1}$ when $F=\left[f_{0}, f_{1}\right]$ is bent.
Proposition 2. (well known)
A (2t)-boolean function $F=\left[f_{0}, f_{1}\right]$ is a bent if and only if:
(a) $f_{0}$ and $f_{1}$ are near-bent.
(b) $\forall u \in \mathbb{F}_{2^{2 t-1}}\left|\hat{f}_{0}(u)\right|+\left|\hat{f}_{1}(u)\right|=2^{t}$

Proof. A proof is given in [9],Proposition 14.
Remark: (b) means that one of $\left|\hat{f}_{0}(a)\right|$ and $\left|\hat{f}_{1}(a)\right|$ is equal to $2^{t}$ and the other one is equal to 0 .

### 1.4. The Kerdock code. :

## Notation:

$Q(x)=\sum_{j=1}^{t-1} \operatorname{tr}\left(x^{2^{j}+1}\right)$.
If $e \in \mathbb{F}_{2^{m}}: t_{e}(x)=\operatorname{tr}(e x), Q_{e}(x)=Q(e x)$.

## Definition 3.

The Kerdock code of length $2^{2 t}$ is the set of the representative vectors of the $2 t$-boolean functions

$$
F=\left[Q_{u}, Q_{u}+t_{u}\right]+\text { affine-linear form }
$$

with $u \in \mathbb{F}_{2^{2 t-1}}$.
Example: The representative vector of $F=\left[Q_{u}, Q_{u}+t_{u}\right]$ is:
$m=\left(f_{0}(0) \ldots \ldots f_{0}\left(\alpha^{i}\right) \ldots \ldots f_{0}\left(\alpha^{n / 2-2}\right) \quad f_{1}(0) \ldots \ldots f_{1}\left(\alpha^{i}\right) \ldots . . f_{1}\left(\alpha^{n / 2-2}\right)\right)$
with $f_{0}(x)=Q_{u}(x)$ and $f_{1}(x)=\left(Q_{u}+t_{u}\right)(x)$
Theorem 4. (Kerdock)
With the above notations:
If $u \neq 0$ then:
$F=\left[Q_{u}, Q_{u}+t_{u}\right]+$ affine-linear form is a Bent Function.
Proof. See [3] or [5] chapter 15.

## Remark:

From the elementary properties of bent functions, $F=\left[Q_{u}, Q_{u}+t_{u}\right]+$ affine-linear form is bent if and only if $\left[Q_{u}, Q_{u}+t_{u}\right.$ ] is bent. Hence we restrict our research to $\left[Q_{u}, Q_{u}+t_{u}\right]$.
Definition: For the sequel of the paper $\left[Q_{u}, Q_{u}+t_{u}\right]$ is called a Kerdock bent function.

The Kerdock code $K_{2 t}$ is a binary non-linear code with several interesting properties. For instance:

1) $\left[Q_{u}, Q_{u}+t_{u}\right]$ is a bent functions
2) $\left[Q_{u}, Q_{u}+t_{u}\right]+\left[Q_{v}, Q_{v}+t_{v}\right]$ is a bent function.

## A problem:

The question of this paper is to replace $t_{u}$ in $\left[Q_{u}, Q_{u}+t_{u}\right]$ by another $2 t-1$-boolean function, for example $t_{r}$, to get another bent function.

### 1.5. Main tools.

## Definition 5.

If $f$ is a $2 t-1)$-near-bent function then $\hat{I}_{f}$ is the indicator of the set $\left\{x \in \mathbb{F}_{2^{2 t-1}} \mid \hat{f}(x)=0\right\}$ where $\hat{f}$ is the Fourier transform of $f$.
(In other words, $\hat{I}_{f}(x)=1$ if and only if $\hat{f}(x)=0$ ).
The two Theorems below are the main tools of the present work.

Theorem 6. (McGuire and Leander)
Let $f$ be a $(2 t-1)$-near-bent function and let $v$ be in $\mathbb{F}_{2^{2 t-1}}$.
$D_{v}\left(\hat{I}_{f}\right)=1$ if and only if $\left[f, f+t_{v}\right]$ is a bent-function.
Proof. See [4], Theorem 3.
Theorem 7. (W)
Let $f$ be a $2 t-1)$-near-bent function.
Let $\omega$ be in $\mathbb{F}_{2^{2 t-1}}$ and let $\epsilon$ be in $\mathbb{F}_{2}$.
If $D_{\omega} f=\epsilon$ then $\hat{I}_{f}=t_{\omega}+\epsilon$.
Remark: According to the definition of $\hat{I}_{f}$ this lemma means that if $D_{\omega} f=\epsilon$ then $\hat{f}(x)=0$ if and only if $t_{\omega}(x)=1+\epsilon$.

Proof. $\hat{f}(u)=\sum_{x \in \mathbb{F}_{2^{2 t-1}}}(-1)^{f(x)+\operatorname{tr}(u x)}=2^{2 t-1}-2 w(f+\operatorname{tr}(u x)$.
$\hat{f}(u)=0$ if and only if $w\left(f+t_{u}\right)=2^{2 t-2}$.
$D_{\omega} f=\epsilon$ means that $f(x+\omega)=f(x)+\epsilon$.
The transform $\tau: x \rightarrow x+\omega$ is a permutation of $\mathbb{F}_{2^{2 t-1}}$ and then preserves the weight of every $(2 t-1)$-Boolean function. Thus:

$$
\sharp\{x \mid f(x)+\operatorname{tr}(u x)=1\}=\sharp\{x \mid f(x+\omega)+\operatorname{tr}(u(x+\omega))=1\} .
$$

$(E) \sharp\{x \mid f(x)+\operatorname{tr}(u x)=1\}=\sharp\{x \mid f(x)+\epsilon+\operatorname{tr}(u x)+\operatorname{tr}(u \omega)=1\}$.
Now assume $\operatorname{tr}(u \omega)+\epsilon=1$. The right hand member of $(E)$ is:

$$
\sharp\{x \mid f(x)+\operatorname{tr}(u x)=0\}=2^{2 t-1}-\sharp\{x \mid f(x)+\operatorname{tr}(u x)=1\}
$$

Hence ( $E$ ) becomes:
$\sharp\{x \mid f(x)+\operatorname{tr}(u x)=1\}=2^{2 t-1}-\sharp\{x \mid f(x)+\operatorname{tr}(u x)=1\}$
In other words $w\left(f+t_{u}\right)=2^{2 t-1}-w\left(f+t_{u}\right)$ and thus:
Conclusion:
If $\operatorname{tr}(u \omega)+\epsilon=1$ then $w\left(f+t_{u}\right)=2^{2 t-2}$ which is equivalent to $\hat{f}(u)=0$.
For every $\epsilon$ the number of $u$ such that $\operatorname{tr}(u \omega)+\epsilon=1$ is $2^{2 t-2}$. This is also the number of $u$ such that $\hat{f}(u)=0$ (see Proposition 1). Then, immediately: $\hat{f}(u)=0$ if and only if $\operatorname{tr}(u \omega)+\epsilon=1$.
This means $\hat{I}_{f}=t_{\omega}+\epsilon$.

## 2. RESULTS

The goal is to find all the $r$ such that $\left[Q_{u}, Q_{u}+t_{r}\right]$ is bent or such that $\left[Q_{u}+Q_{v}, Q_{u}+Q_{v}+t_{r}\right]$ is bent.
Strategy:
For $f=Q_{u}$ or $f=Q_{u}+Q_{u}$, in order to apply Theorem 5(McGuire and Leander) we have to find $\hat{I}_{f}$ and $D_{r}\left(\hat{I}_{f}\right)$.
2.1. The case. $\left[Q_{u}, Q_{u}+t_{r}\right]$.

## Theorem 8.

$$
\text { If } f=Q_{u} \text { then } \hat{I}_{f}=\epsilon+t_{u^{-1}} \text { with } \epsilon \in \mathbb{F}_{2} \text {. }
$$

Proof.
$Q_{u}(x)=\sum_{j=1}^{t-1} \operatorname{tr}\left((u x)^{2^{j}+1}\right)$.
If $f_{j}(x)=(u x)^{2^{j}+1}$ then $D_{u^{-1}} f_{j}(x)=(u x)^{2^{j}+1}+\left[u\left(x+u^{-1}\right)\right]^{2^{j}+1}$

$$
=u x+u^{2^{j}} x^{2^{j}}+1
$$

$\operatorname{tr}\left(f_{j}(x)\right)=\operatorname{tr}(x)+\operatorname{tr}\left(u^{2^{j}} x^{2^{j}}\right)+\operatorname{tr}(1)=\operatorname{tr}(1)=1$
$D_{u^{-1}} Q_{u}(x)=\sum_{j=1}^{t-1} D_{u^{-1}} f_{j}(x)=\sum_{j=1}^{t-1} 1=t-1=\epsilon \in \mathbb{F}_{2}$.
According to the previous theorem: $\hat{I}_{Q_{u}}=t_{u^{-1}}+\epsilon$.
Theorem 9. Let $u$ and $r$ be in $\mathbb{F}_{2^{2 t-1}}$.

$$
\left[Q_{u}, Q_{u}+t_{r}\right] \text { is bent if and only if } \operatorname{tr}\left(u^{-1} r\right)=1 .
$$

Proof.

$$
\begin{aligned}
D_{r}\left(\hat{\hat{I}_{Q u}}\right)(x) & =\operatorname{tr}\left(u^{-1} x\right)+\epsilon+\operatorname{tr}\left(u^{-1}(x+r)\right)+\epsilon \\
& =\operatorname{tr}\left(u^{-1} x\right)+\operatorname{tr}\left(u^{-1} x\right)+\operatorname{tr}\left(u^{-1} r\right) \\
& =\operatorname{tr}\left(u^{-1} r\right) .
\end{aligned}
$$

Then, from McGuire and Leander:
$\left[Q_{u}, Q_{u}+t_{r}\right]$ is bent if and only if $\operatorname{tr}\left(u^{-1} r\right)=1$.
2.2. The case. $\left[Q_{u}+Q_{v}, Q_{u}+Q_{v}+t_{r}\right]$

Under the assumption on $u$ and $v$ then $\left[Q_{u}+Q_{v}, Q_{u}+Q_{v}+t_{u}+t_{v}\right]$ is a bent function. See Theorem 4, 5)
Hence $f(x)=Q_{u}+Q_{v}$ is near-bent (proposition 3).
Now we search $\omega \in F_{2^{2 t-1}}$ such that $D_{\omega} f=\epsilon$ with $\epsilon \in \mathbb{F}_{2}$.
$D_{\omega} f=D_{\omega} Q_{u}+D_{\omega} Q_{v}$.
$Q_{u}(x)=\sum_{j=1}^{t-1} \operatorname{tr}\left[f_{u, j}(x)\right]$ with $f_{u, j}(x)=(u x)^{2^{j}+1}$.
Since $D_{\omega}, \sum, \operatorname{tr}$ are additive functions then:

$$
\begin{aligned}
& D_{\omega} Q_{u}=\sum_{j=1}^{t-1} \operatorname{tr}\left[D_{\omega} f_{u, j}\right] . \\
& \begin{aligned}
& D_{\omega} f_{u, j}(x)=u^{2^{j}+1} x^{2^{j}+1}+u^{2^{j}+1}(x+\omega)^{2^{j}+1} . \\
& \begin{aligned}
(x+\omega)^{2^{j}+1} & =(x+\omega)^{2^{j}}(x+\omega)=\left(x^{2^{j}}+\omega^{2^{j}}\right)(x+\omega) . \\
\quad & =x^{2^{j}+1}+\omega^{2^{j}} x+\omega x^{2^{j}}+\omega^{2^{j}+1} .
\end{aligned} \\
&\left.\begin{array}{c}
D_{\omega} f_{u, j}(x)
\end{array}\right) u^{2^{j}+1}\left(\omega^{2^{j}} x+\omega x^{2^{j}}+\omega^{2^{j}+1}\right) .
\end{aligned} \\
& \begin{aligned}
& D_{\omega} Q_{u}(x)=\sum_{j=1}^{t-1} \operatorname{tr}\left[u^{2^{j}+1}\left(\omega^{2^{j}} x+\omega x^{2^{j}}+\omega^{2^{j}+1}\right)\right] . \\
&\left.\left.\quad=\sum_{j=1}^{t-1} \operatorname{tr}\left[u(u \omega)^{2 j} x\right]+\sum_{j=1}^{t-1} \operatorname{tr}\left[u^{2^{j}+1} \omega x^{2^{j}}\right)\right]+\sum_{j=1}^{t-1} \operatorname{tr}\left[u^{2^{j}+1} \omega^{2^{j}+1}\right)\right] .
\end{aligned}
\end{aligned}
$$

With $m=2 t-1$ and since $x^{m}=x$ and $u^{m}=u$ :
$\hat{f}(u)=\sum_{x \in \mathbb{F}_{2} 2 t-1}(-1)^{f(x)+\operatorname{tr}(u x)}=2^{2 t-1}-2 w\left(f+\operatorname{tr}(u x) . \sum_{j=1}^{t-1} \operatorname{tr}\left[u^{2^{j}+1} \omega x^{2^{j}}\right)\right]=$
$\sum_{j=1}^{t-1} \operatorname{tr}\left[\left(u^{2^{j}+1} \omega x^{2^{j}}\right)^{2^{m-j}}\right]=\sum_{j=1}^{t-1} \operatorname{tr}\left[u(u \omega)^{2^{m-j}} x\right]$.
and thus:
$\left.D_{\omega} Q_{u}(x)=\sum_{j=1}^{t-1} \operatorname{tr}\left[u(u \omega)^{2^{j}} x\right]+\sum_{j=1}^{t-1} \operatorname{tr}\left[u(u \omega)^{2^{m-j}} x\right]+\sum_{j=1}^{t-1} \operatorname{tr}\left[u^{2^{j}+1} \omega^{2^{j}+1}\right)\right]$.
When $j$ runs from 1 to $t-1$ then $m-j$ runs from $2 t-2$ to $t$.
Hence: $D_{\omega} Q_{u}=\sum_{j=1}^{t-1} \operatorname{tr}\left[D_{\omega} f_{u, j}\right]$.
$\left.D_{\omega} Q_{u}(x)=\operatorname{tr}\left[u \sum_{j=1}^{2 t-2}(u \omega)^{2^{j}}\right] x+\sum_{j=1}^{t-1} \operatorname{tr}\left[u^{2^{j}+1} \omega^{2^{j}+1}\right)\right]$.
By replacing $u$ by $v$ we find a similar result and finally:
$D_{\omega} f(x)=\operatorname{tr}\left(\left[u \sum_{j=1}^{2 t-2}(u \omega)^{2^{j}}+v \sum_{j=1}^{2 t-2}(v \omega)^{2^{j}}\right] x\right)+\epsilon$ with $\epsilon \in \mathbb{F}_{2}$.
$\hat{f}(u)=\sum_{x \in \mathbb{F}_{2} 2 t-1}(-1)^{f(x)+\operatorname{tr}(u x)}=2^{2 t-1}-2 w(f+\operatorname{tr}(u x)$. It follows that $D_{\omega} f$ is a constant function if and only if
(*) $u \sum_{j=1}^{2 t-2}(u \omega)^{2^{j}}+v \sum_{j=1}^{2 t-2}(v \omega)^{2^{j}}=0$.
Remark that $\sum_{j=1}^{2 t-2}(u \omega)^{2^{j}}=u \omega+\operatorname{tr}(u \omega)$ and $\sum_{j=1}^{2 t-2}(v \omega)^{2^{j}}=v \omega+\operatorname{tr}(v \omega)$. Then $(*)$ becomes:
$(*) \quad\left(u^{2}+v^{2}\right) \omega+u \operatorname{tr}(u \omega)+v \operatorname{tr}(v \omega)=0$.
Case 1: $\operatorname{tr}(u \omega)=\operatorname{tr}(v \omega)=0$.
we find the trivial solution $\omega=0$.
Case 2: $\operatorname{tr}(u \omega)=\operatorname{tr}(v \omega)=1$.
$\omega=(u+v)^{-1}$ and $\operatorname{tr}(u \omega)=\operatorname{tr}\left[u(u+v)^{-1}\right], \operatorname{tr}(v \omega)=\operatorname{tr}\left[v(u+v)^{-1}\right]$.
This leads to $\operatorname{tr}(u \omega)+\operatorname{tr}(v \omega)=\operatorname{tr}\left[(u+v)(u+v)^{-1}\right]=\operatorname{tr}(1)=1$ if $\operatorname{tr}\left(u^{-1} r\right)=1$.
which is imposible because $\operatorname{tr}(u \omega)=\operatorname{tr}(v \omega)$.
Case 3: $\operatorname{tr}(u \omega)=1, \operatorname{tr}(v \omega)=0$,
$D_{\omega} Q_{u}=\sum_{j=1}^{t-1} \operatorname{tr}\left[D_{\omega} f_{u, j}\right] . \omega=u\left(u^{2}+v^{2}\right)^{-1}$
Case 4: $\operatorname{tr}(u \omega)=0, \operatorname{tr}(v \omega)=1$.
$\omega=v\left(u^{2}+v^{2}\right)^{-1}$.
In case $3, \operatorname{tr}(u \omega)=\operatorname{tr}\left(u^{2}\left(u^{2}+v^{2}\right)^{-1}\right)=\operatorname{tr}\left[\left(u(u+v)^{-1}\right)^{2}\right]$.
$=\operatorname{tr}\left[u(u+v)^{-1}\right]$. Similarly in case 4:
$\operatorname{tr}(v \omega)=\operatorname{tr}\left[v(u+v)^{-1}\right]$. Then $\operatorname{tr}\left[u(u+v)^{-1}\right]=\operatorname{tr}\left[v(u+v)^{-1}\right]$ is impossible since
$\operatorname{tr}\left(u^{-1} r\right)=1 . \operatorname{tr}\left[u(u+v)^{-1}\right]+\operatorname{tr}\left[v(u+v)^{-1}\right]=\operatorname{tr}\left((u+v)(u+v)^{-1}\right)=$ $\operatorname{tr}(1)=1$. Conclusion:
Proposition 10. $f=Q_{u}+Q_{v}$.
If $\omega$ is a non-zero element such that $D_{\omega} f=\epsilon$ with $\epsilon \in \mathbb{F}_{2}$ then:

$$
\begin{aligned}
& \omega=u\left(u^{2}+v^{2}\right)^{-1} \text { if } \operatorname{tr}\left[u(u+v)^{-1}\right]=1 . \\
& \omega=v\left(u^{2}+v^{2}\right)^{-1} \text { if } \operatorname{tr}\left[v(u+v)^{-1}\right]=1 .
\end{aligned}
$$

We are now in position to find all $e \in \mathbb{F}_{2^{2 t-1}}$ such that $\left[f, f+t_{e}\right]$ is a bent function and consider the case $e=r+s$

## Theorem 11.

Let $u \neq 0, \operatorname{tr}\left(u^{-1} r\right)=1, v \neq 0, \operatorname{tr}\left(v^{-1} s\right)=1, u \neq v, r \neq s$.
Define $\omega$ by:

$$
\begin{aligned}
& \omega=u\left(u^{2}+v^{2}\right)^{-1} \text { if } \operatorname{tr}\left(u(u+v)^{-1}\right)=1 . \\
& \omega=v\left(u^{2}+v^{2}\right)^{-1} \text { if } \operatorname{tr}\left(v(u+v)^{-1}\right)=1 .
\end{aligned}
$$

If $\operatorname{tr}(\omega(r+s))=1$ then:

$$
\left[Q_{u}, t_{r}\right]+\left[Q_{v}, t_{s}\right]
$$

is a bent function.
Proof.
Applying Theorem 7, since $D_{\omega} f=\epsilon$ then $\hat{I}_{f}=t_{\omega}+\epsilon$. Now, if $e \in \mathbb{F}_{2^{2 t-1}}$ then $D_{e} \hat{I}_{f}(x)=\hat{I}_{f}(x)+\hat{I}_{f}(x+e)=\operatorname{tr}(\omega x+\operatorname{tr}(\omega(x+e)=\operatorname{tr}(\omega x)+$ $\operatorname{tr}(\omega x)+\operatorname{tr}(\omega e)=\operatorname{tr}(\omega e)$. Hence $D_{e} \hat{I}_{f}(x)=1$ if and only if $\operatorname{tr}(\omega e)=1$. From Theorem $5,\left[f, f+t_{e}\right]$ is a bent function if and only if $\operatorname{tr}(\omega e)=1$. Now if $f=Q_{u}+Q_{v}$ then $\left[f, f+t_{r+s}\right]=\left[Q_{u}, Q_{u}+r\right]+\left[Q_{v}, Q_{v}+s\right]$ is a bent function if and only if $\operatorname{tr}(\omega(r+s))=1$.

## 3. Another construction.

## Theorem 12.

Let $\gamma$ be in $\mathbb{F}_{2^{2 t-1}}, \operatorname{tr}\left(u^{-1} r\right)=1$ then:
$\left[Q_{u}+t_{1} t_{\gamma}, Q_{u}+t_{r}+t_{1} t_{\gamma}\right]$ is a bent function.
Proof.
This is a special case of Theorem 20 of [10] with $f_{0}=Q_{u}$ and $f_{1}=Q_{u}+t_{r}$

Examples:
$\left[Q_{u}+t_{1} t_{\gamma}, Q_{u}+t_{r}+t_{1} t_{\gamma}\right]$ with $\operatorname{tr}\left(u^{-1} r\right)=1$.
$\left[Q_{u}+Q_{v}+t_{1} t_{\gamma}, Q_{u}+Q_{v}+t_{r+s}+t_{1} t_{\gamma}\right]$ with conditions of Theorem 11 on $u, v, r, s$.

## 4. Conclusions

By using a slight modification of Kerdock bent functions we have introduced new bent functions.

The number of new bent functions $\left[Q_{u}, Q_{u}+t_{r}\right]$ (Theorem 9) is greater than the number of Kerdock bent functions $\left[Q_{u}, Q_{u}+t_{u}\right]$,.
The number of new bent functions $\left[Q_{u}+t_{1} t_{\gamma}, Q_{u}+t_{r}+t_{1} t_{\gamma}\right]$, $u \neq 0, \operatorname{tr}\left(u^{-1} r\right)=1, \gamma \neq 0$ (Theorem 12) is greater than the number of Kerdock bent functions $\left[Q_{u}, Q_{u}+t_{r}\right.$ ].
It is easy to check that:

Bent Functions

$$
\begin{aligned}
& {\left[Q_{u}, Q_{u}+t_{u}\right], u \neq 0} \\
& {\left[Q_{u}, Q_{u}+t_{r}\right] \text { with } \operatorname{tr}\left(u^{-1} r\right)=1} \\
& {\left[Q_{u}+t_{1} t_{\gamma}, Q_{u}+t_{r}+t_{1} t_{\gamma}\right]}
\end{aligned}
$$

$$
\left[Q_{u}, Q_{u}+t_{u}\right]+\left[Q_{v}, Q_{v}+t_{v}\right]
$$

$$
\left[Q_{u}, Q_{u}+t_{r}\right]+\left[Q_{v}, Q_{v}+t_{s}\right] \quad A\left(2^{2 t-1}-1\right)^{2}
$$

$$
\text { with } A=\sharp\left\{(r, s) \mid \operatorname{tr}\left(u^{-1} r\right)=1, \operatorname{tr}\left(u^{-1} s\right)=1, \operatorname{tr}(\omega(r+s)=1\}\right. \text {. }
$$

(Notations of Theorem 11.)

## 5. References

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