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Abstract

We analyze $1 - d$ forced steady state scalar conservation laws. We first show the existence and uniqueness of entropy solutions as limits as $t \to \infty$ of the corresponding solutions of the scalar evolutionary hyperbolic conservation law. We then linearize the steady-state equation with respect to perturbations of the forcing term. This leads to a linear first order differential equation with, possibly, discontinuous coefficients. We show the existence and uniqueness of solutions in the context of duality solutions. We also show that this system corresponds to the steady state version of the linearized evolutionary hyperbolic conservation law. This analysis leads us to the study of the sensitivity of the shock location with respect to variations of the forcing term, an issue that is relevant in applications to optimal control and parameter identification problems.

1. Introduction

Optimal control of solutions to (non-linear) hyperbolic conservation laws is hampered by the presence of discontinuities (shock waves) making the use of the standard techniques based on linearization rather delicate. The same can be said about optimal design and parameter identification problems. For an analysis of these issues in the context of evolutionary hyperbolic conservation laws in $1 - d$ we refer to [7] and [8] and the references therein. The issue of the convergence of time-evolution controls towards the steady state ones as the time horizon tends to infinity is a subject that recently has attracted attention (see, for instance, [6] and [14]). But, as far as we know, this has not yet been addressed in the context of the scalar conservation laws considered here.

In this paper, we consider a simplified model problem based on the $1 - d$ *scalar steady driven conservation law*

$$\partial_x f(v(x)) + v(x) = g(x), \ x \in \mathbf{R}, \tag{1.1}$$

supplemented with a "far field" boundary condition

$$\lim_{x \to \pm \infty} v(x) = 0. \tag{1.2}$$

Note that, as we shall see in Section 2 below, v can be viewed as the asymptotic limit for $t \to \infty$ of solutions to the associated evolutionary scalar hyperbolic conservation law

$$\partial_t u(t, x) + \partial_x f(u(t, x)) + u(t, x) = g(x), \ u(0, x) = u^0. \tag{1.3}$$

Our main goal is to perform a rigorous *sensitivity analysis* of solutions v of the steady problem (1.1), (1.2) with respect to perturbations of the forcing term g. In particular, denoting v_ε the solution of (1.1) - (1.2) with $g = g + \varepsilon \delta g$ we identify the limit of the quantity

$$\frac{v_\varepsilon - v}{\varepsilon} = h_\varepsilon \to h \text{ as } \varepsilon \to 0,$$

where v is the solution associated to g.

It turns out that h is in general a measure on $\mathbf{R}$, with a singular part sitting on the set of discontinuities of v, taking account of the sensitivity of the shock location of v, see Section 4. Our approach is based on an adaptation of the concept of duality solutions introduced by Bouchut and James [2, 3] and Strömberg [15].

The paper is organized as follows. In Section 2, we introduce the concept of entropy solution to problems (1.1) and (1.3) and discuss its basic properties. In particular, we recall the standard result on exponential stability of the stationary solution v as $t \to \infty$. In Section 3, we discuss the properties of stationary solutions of (1.1) and, in particular, the structure of the set of singularities. Section 4 is devoted to the analysis of stability issues by means of a suitable linearization of the stationary problem (1.1) - (1.2). Possible extensions and further discussions can be found in Section 5.

2. The asymptotic limit as $t \to \infty$

This section is devoted to analysis of the models under consideration from close, using the notion of entropy solutions. Of course, our analysis can be viewed as a particular instance of the general theory of nonlinear semigroups in $L^1(\mathbf{R})$ (see [9]). The steady problem and the evolution one are linked in the sense that the latter can be solved in a unique manner by the semigroup of $L^1(\mathbf{R})$-contractions generated by the accretive operator associated to the steady state problem.

Throughout the paper we suppose that the flux function f is locally Lipschitz,

$$f \in W^{1,\infty}_{\mathrm{loc}}(\mathbf{R}). \tag{2.1}$$

We start with the nowadays standard definition of *entropy solution* to the evolutionary problem (1.3) introduced by Kružkov [10], [11] and [12].

Definition 2.1. *Let $g \in L^\infty(\mathbf{R})$. A function $u = u(t, x)$,*

$$u \in L^\infty((0, T) \times \mathbf{R}) \cap C([0, T]; L^1_{\mathrm{loc}}(\mathbf{R})),$$

is an entropy solution *to problem (1.3) if it solves the equation in the sense of distributions, and, moreover, fulfills the integral inequality*

$$\int_{\mathbf{R}} E\left(u(\tau, \cdot)\right) \varphi(\tau, \cdot) \mathrm{d}x - \int_{\mathbf{R}} E\left(u^0\right) \varphi(0, \cdot) \, \mathrm{d}x$$

$$\leq \int_0^\tau \int_{\mathbf{R}} \left(E(u)\partial_t \varphi + F(u)\partial_x \varphi - E'(u)u\varphi + E'(u)g\varphi \right) \mathrm{d}x \, \mathrm{d}t, \tag{2.2}$$

for any $\tau \in [0, T]$, any test function $\varphi \in C_c^\infty([0, T] \times \mathbf{R})$, $\varphi \geq 0$, and any convex entropy $E : \mathbf{R} \to \mathbf{R}$ with $F' \equiv E'f'$.

Note that the *existence* of global-in-time entropy solutions for $u^0 \in L^\infty(\mathbf{R})$ can be established by means of artificial viscosity approximations (see Kružkov [12]).

It can also be shown that two entropy solutions u^1, u^2 emanating from the initial data u_0^1, u_0^2 satisfy

$$\int_{|x| \leq M} \left| u^1(\tau, \cdot) - u^2(\tau, \cdot) \right| \, \mathrm{d}x + \int_0^\tau \int_{|x| \leq M + \lambda(\tau - t)} \left| u^1 - u^2 \right| \, \mathrm{d}x \, \mathrm{d}t$$

$$\leq \int_{|x| \leq M + \lambda \tau} \left| u_0^1 - u_0^2 \right| \, \mathrm{d}x \tag{2.3}$$

for any $\tau \geq 0$, $M > 0$, where

$$\lambda = \sup \left\{ |f'(z)| \; \middle| \; |z| \leq (1 + \tau) \max \left\{ \|u_0^1\|_{L^\infty(\mathbf{R})}, \|u_0^2\|_{L^\infty(\mathbf{R})}, \|g\|_{L^\infty(\mathbf{R})} \right\} \right\}. \tag{2.4}$$

Consequently,

$$\int_{\mathbf{R}} \left| u^1(\tau, \cdot) - u^2(\tau, \cdot) \right| \, \mathrm{d}x + \int_0^\tau \int_{\mathbf{R}} \left| u^1 - u^2 \right| \, \mathrm{d}x \, \mathrm{d}t \leq \int_{\mathbf{R}} \left| u_0^1 - u_0^2 \right| \, \mathrm{d}x. \tag{2.5}$$

In particular, the entropy solutions are uniquely determined by the initial data.

As an immediate consequence of the previous estimate we deduce the exponential stability of entropy steady state solutions.

Corollary 2.1. *Let the flux f be a locally Lipschitz function on $\mathbf{R}$. Let $v \in L^\infty(\mathbf{R})$ be an entropy solution to the stationary problem (1.1) and u an entropy solution to the evolutionary problem (1.3), with $u(0, \cdot) = u^0 \in L^\infty(\mathbf{R})$, such that*

$$\|u^0 - v\|_{L^1(\mathbf{R})} < \infty.$$

Then

$$\|u(t, \cdot) - v\|_{L^1(\mathbf{R})} \leq \exp(-t)\|u^0 - v\|_{L^1(\mathbf{R})} \text{ for any } t \geq 0.$$

Proof: In accordance with the previous discussion, we have

$$\int_{\mathbf{R}} |u(\tau, x) - v(x)| \, \mathrm{d}x + \int_0^\tau \int_{\mathbf{R}} |u(t, x) - v(x)| \, \mathrm{d}x \, \mathrm{d}t \leq \int_{\mathbf{R}} \left| u^0(x) - v(x) \right| \, \mathrm{d}x,$$

and, consequently,

$$\|u(t, \cdot) - v\|_{L^1(\mathbf{R})} \leq \exp(-t)\|u^0 - v\|_{L^1(\mathbf{R})}$$

since

$$a(t) = \exp(-t)\|u^0 - v\|_{L^1(\mathbf{R})},$$

where a is the unique solution of the equation

$$a'(t) + a(t) = 0, \; a(0) = \|u^0 - v\|_{L^1(\mathbf{R})}.$$

Remark 2.1. *In Corollary 2.1 the stationary solution v is arbitrary, in particular, it does not need to satisfy the far field condition (1.2) provided $u^0 - v$ belongs to $L^1(\mathbf{R})$.*

The conclusion of Corollary 2.1 remains valid for solutions of general scalar conservation laws in $\mathbf{R}^N$ provided the flux field is locally Lipschitz.

3. Stationary solutions

For the sake of simplicity, we focus on the stationary solutions satisfying the far field condition (1.2). To this end, we restrict ourselves to the class of forcing terms

$$g \in L^1 \cap L^\infty(\mathbf{R}). \tag{3.1}$$

Under these circumstances, in view of (2.5), it is easy to see that the stationary problem (1.1) admits a unique entropy solution v in the same class, namely,

$$v \in L^1 \cap L^\infty(\mathbf{R}). \tag{3.2}$$

In addition, by the conservation of mass we have

$$\int_{\mathbf{R}} \left(v^1 - v^2 \right) \, \mathrm{d}x = \int_{\mathbf{R}} \left(g^1 - g^2 \right) \, \mathrm{d}x, \tag{3.3}$$

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the comparison principle

$$\int_{\mathbf{R}} [v^1 - v^2]^+ \ \mathrm{d}x \le \int_{\mathbf{R}} [g^1 - g^2]^+ \ \mathrm{d}x \tag{3.4}$$

and the L^1-contraction property

$$\int_{\mathbf{R}} |v^1 - v^2| \ \mathrm{d}x \le \int_{\mathbf{R}} |g^1 - g^2| \ \mathrm{d}x \ , \tag{3.5}$$

where v^1, v^2 are the entropy solutions of (1.1) with $g = g^1$, $g = g^2$, respectively.

The existence of stationary solutions out of (2.5) can be proved in several different ways. For instance, using the L^1-distance as Lyapunov function and LaSalle's invariance principle (see [16]) one can show that there is an unique stationary solution and that all other solutions converge exponentially as $t \to \infty$ to it. One can also construct the stationary solutions as limits of time periodic solutions of time-period $\tau > 0$ with $\tau \to 0$. These periodic solutions can be built as fixed points (using Banach contraction principle) of the semigroup map associating the value of the solution at time $t = \tau$ to the initial datum. Furthermore, one can also build the stationary solutions as vanishing viscosity limits of elliptic equations. In fact, as mentioned above, these results are also a particular example of the classical theory of nonlinear contraction semigroups in $L^1(\mathbf{R})$.

The main result of this section is as follows:

Proposition 3.1. *Let the flux function f be continuously differentiable on $\mathbf{R}$ and non-degenerate in the sense that the critical points $f'(y) = 0$ are isolated in $\mathbf{R}$. Let*

$$g \in L^1 \cap L^\infty \cap BV(\mathbf{R}).$$

Then the stationary problem (1.1) possesses an entropy solution v determined uniquely in the class

$$v \in L^1 \cap L^\infty(\mathbf{R})$$

such that v is continuous in $\mathbf{R}$ with a possible exception of a countable set of points $\{s_i\}$, with

$$\lim_{x \to s_i-} v(x) = v^i_- \ne v^i_+ = \lim_{x \to s_i+} v(x), \ i = 1, \dots, N.$$

Each open interval with end points v^i_-, v^i_+ contains at least one critical point y^i of f such that either

$$f(y^i) < f(v^i_-) = f(v^i_+) \ yielding \ v^i_- > v^i_+$$

or

$$f(y^i) > f(v^i_-) = f(v^i_+) \ yielding \ v^i_- < v^i_+.$$

Proof: Assuming, in addition to (3.1), that

$$g \in BV(\mathbf{R}), \tag{3.6}$$

(3.5) yields immediately

$$v \in BV(\mathbf{R}). \tag{3.7}$$

Identifying v with its Lebesgue means,

$$v(x) = \lim_{h \to 0} \frac{1}{2h} \int_{x-h}^{x+h} v(z) \ \mathrm{d}z,$$

we deduce that v is continuous with a possible exception of countably many points at which the left and right limits exist.

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Assume that $x_0 \in \mathbf{R}$ is a point of discontinuity of v, specifically,

$$\lim_{x \to x_0-} v(x) = v_- \neq v_+ = \lim_{x \to x_0+} v(x),$$

where, as $f(v)$ is Lipschitz continuous (since $\partial_x(f(v)) = g - v$),

$$f(v-) = f(v_+). \tag{3.8}$$

Note that this null jump condition is the natural limit of the Rankine-Hugoniot condition for the damped evolutionary hyperbolic conservation law (1.3) or, in other words, the condition characterizing stationary shocks.

Since v is an entropy solution of (1.1) we deduce that

$$\partial_x\left[(f(v) - f(k))\mathrm{sgn}(v - k)\right] \leq c \text{ in the sense of distributions,}$$

for some finite c. This can be easily seen by the vanishing viscosity argument, or directly from (2.2) in view of the fact that both g and v are bounded, for instance. In particular,

$$\lim_{x \to x_0-} (f(v) - f(k))\,\mathrm{sgn}(v - k)$$

$$= (f(v_-) - f(k))\,\mathrm{sgn}(v_- - k) \geq (f(v_+) - f(k))\,\mathrm{sgn}(v_+ - k)$$

$$= \lim_{x \to x_0+} (f(v) - f(k))\,\mathrm{sgn}(v - k) \tag{3.9}$$

provided k belongs to the open interval I with the end points v_-, v_+.

We distinguish three complementary cases:

- $f(k) < f(v_-) = f(v_+)$ for some k in the interval linking v_- and v_+. In this case, relation (3.9) implies

$$v_- > v_+.$$

- $f(k) > f(v_-) = f(v_+)$ for some k in the interval linking v_- and v_+. Similarly, we deduce from (3.9) that

$$v_+ > v_-.$$

- The degenerate case in which $f(v-) = f(k) = f(v_+)$ can be excluded by the main assumptions of the Proposition in which we impose the set of critical points of f to be isolated.

4. Sensitivity with respect to the forcing term

In this section, we study the sensitivity of solutions to the stationary problem (1.1), (1.2) with respect to perturbations of the right-hand side g. To this end, we introduce the concept of *duality solutions* in the spirit of Bouchut and James [3].

4.1. Duality solutions for the stationary problem

We start with a prototype example of a scalar conservation law, where the flux function f is strictly convex with a (global) minimum attained in $\mathbf{R}$. By virtue of Proposition 3.1, solutions of problem (1.1) admit a countable (possibly empty) set of singularities where the solutions "jump down" across the shock, meaning they satisfy the so-called *Oleinik condition*. Note that, in particular, the degenerate case can be excluded because of the strict convexity assumption on f.

Furthermore, by comparison (the maximum principle), it can be shown that

$$\partial_x v \leq \overline{g} \text{ in } \mathcal{D}'(\mathbf{R}) \tag{4.1}$$

provided that
$$\partial_x g \le \overline{g} \text{ in } \mathcal{D}'(\mathbf{R}). \tag{4.2}$$

We now study the effect of a perturbation
$$g_\varepsilon = g + \varepsilon(\delta g),$$

with
$$\delta g \in L^1 \cap L^\infty \cap BV(\mathbf{R})$$
$$\delta g \quad \text{of compact support in } \mathbf{R},$$
$$\partial_x \delta g \le \overline{\delta g} \text{ in } \mathrm{D}'(\mathbf{R}).$$

This leads to an entropy solution $v_\varepsilon \in L^1 \cap L^\infty(\mathbf{R})$ of (1.1) that can be written in the form
$$v_\varepsilon = v + \varepsilon(\delta v)_\varepsilon,$$

more specifically,
$$\partial_x f\left([v + \varepsilon(\delta v)_\varepsilon]\right) + [v + \varepsilon(\delta v)_\varepsilon] = g + \varepsilon(\delta g). \tag{4.3}$$

Note that, since we deal with entropy solutions,
$$\|(\delta v)_\varepsilon\|_{L^1(\mathbf{R})} \le \|\delta g\|_{L^1(\mathbf{R})}. \tag{4.4}$$

Furthermore, we have
$$\partial_x \left[\frac{1}{\varepsilon}\left(\int_0^\varepsilon f'(v + z(\delta v)_\varepsilon)\ \mathrm{d}z\right)(\delta v)_\varepsilon\right] + (\delta v)_\varepsilon = \delta g; \tag{4.5}$$

whence the perturbation $h_\epsilon = (\delta v)_\varepsilon$ solves the *linear problem*:
$$\partial_x\left(A_\varepsilon h_\varepsilon\right) + h_\varepsilon = \delta g, \tag{4.6}$$

where
$$A_\epsilon = \frac{1}{\varepsilon}\left(\int_0^\varepsilon f'(v + z(\delta v)_\varepsilon)\ \mathrm{d}z\right),$$

$$\|A_\varepsilon\|_{L^\infty(\mathbf{R})} \le c\left(\|g\|_{L^\infty(\mathbf{R})}, \|(\delta g)\|_{L^\infty(\mathbf{R})}\right), \tag{4.7}$$
$$A_\varepsilon \to A \text{ weakly-* in } L^\infty(\mathbf{R}). \tag{4.8}$$

Furthermore, in accordance with (4.1), (4.2),
$$\partial_x A_\varepsilon = \frac{1}{\varepsilon}\int_0^\varepsilon \left[f''\left(v + z(\delta v)_\varepsilon\right)\partial_x\left(v + z(\delta v)_\varepsilon\right)\right]\ \mathrm{d}z \le c(f)(\overline{g} + \overline{\delta g}), \tag{4.9}$$

with
$$\partial_x g \le \overline{g}, \ \partial_x(\delta g) \le \overline{\delta g}.$$

This yields local uniform BV-bounds on A_ϵ. These arguments, together with the continuity of f' allow us to deduce that
$$A = f'(v). \tag{4.10}$$

As a consequence we expect the limit measure h to be a solution of
$$\partial_x\left(f'(v)h\right) + h = \delta g. \tag{4.11}$$

Note however that $f'(v)$ may be discontinuous at the shock discontinuities of v. Thus, we need to introduce a suitable concept of weak solution in the class of measures in order to determine the solution h in a unique manner and to justify the limit process above. For doing that, the fact that the coefficients involved in the linearized equation fulfill a one-sided Lipschitz condition of the form (4.9) will play a key role.

Definition 4.1. *We say that h is a* duality solution *of*

$$\partial_x \left(Ah \right) + h = \delta g \tag{4.12}$$

if

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbf{R}} ph \, \mathrm{d}x + \int_{\mathbf{R}} ph \, \mathrm{d}x = - \int_{\mathbf{R}} (\delta g) p \, \mathrm{d}x \tag{4.13}$$

for any reversible solution p of the evolutionary problem

$$\partial_t p + A \partial_x p = 0 \tag{4.14}$$

in the sense of Bouchut and James [3].

Remark 4.1.

- *In [3] reversible solutions of the adjoint equation (4.14) are built as limits of approximating sequences of solutions of a final value problem at $t = T$ for regularized potentials A. In [3] it is shown that, under the one-sided Lipschitz condition on A, this leads to a well identified unique solution, the so called reversible one.*

- *Typically, as shown in [7], when linearizing scalar conservation laws along a solution with a shock discontinuity, the adjoint system of the linearized one has the structure (4.14) with the potential A being discontinuous along the shock. The corresponding solution p can be defined by characteristics away from the zone of influence of the shock. The reversible solution is that taking over this set the corresponding value at the shock location at the final time $t = T$. In this way, for locally Lipschitz continuous data at time $t = T$ reversible solutions are locally Lipschitz.*

  Accordingly, the duality formula (4.13) makes sense for h a measure.

- *Note that duality solutions of (4.12) are uniquely determined by the right-hand side δg in the class of measures $h \in \mathcal{M}(\mathbf{R})$.*

Seeing that, by virtue of (4.9), the partial derivatives $\partial_x A_\varepsilon$ are bounded from above so that they fulfill uniformly the one-sided Lipschitz condition. Then we may use the abstract convergence result of Bouchut and James [3] to conclude that

$$h_\varepsilon \equiv (\delta v)_\varepsilon \to \delta v \text{ weakly-* in } \mathcal{M}(\mathbf{R}),$$

where $\delta v \in \mathcal{M}(\mathbf{R})$ is the unique duality solution of the linearized problem

$$\partial_x \left(f'(v) \delta v \right) + \delta v = \delta g. \tag{4.15}$$

Note that the compactness results in [3] refer to the evolution equation but, accordingly, they can also be applied to steady state solutions. This procedure yields a duality solution to the limit linearized steady-state problem (4.11).

The uniqueness of the duality solution for the steady problem (4.11) is easy to prove. In case there were two distinct solutions, h_1 and h_2, then $h = h_1 - h_2$ would satisfy

$$\frac{d}{dt} \int_{\mathbf{R}} ph \, \mathrm{d}x + \int_{\mathbf{R}} ph \, \mathrm{d}x = 0 \tag{4.16}$$

for all reversible solutions of (4.14). Considering all possible reversible solutions of (4.14) associated to all Lipschitz continuous data $p_T = p_T(x)$ at $t = T$ we would conclude that

$$\int_{\mathbf{R}} p_T h \, \mathrm{d}x = 0,$$

and this would yield $h \equiv 0$.

We have proved the following result.

Theorem 4.1. *Let $f : \mathbf{R} \mapsto \mathbf{R}$ be a strictly convex function in $C^2(\mathbf{R})$. Let*

$$g, \delta g \in L^1 \cap L^\infty \cap BV(\mathbf{R})$$

be given such that

$$\partial_x g \le \overline{g}, \ \partial_x(\delta g) \le \overline{\delta g} \ \text{in } \mathcal{D}'(\mathbf{R}).$$

Let

$$v_\varepsilon = v + \varepsilon(\delta v)_\varepsilon \in L^1 \cap L^\infty(\mathbf{R})$$

be the solution of the perturbed problem (4.3), where $v \in L^1 \cap L^\infty(\mathbf{R})$ is the unique solution of (1.1).
  Then

$$(\delta v)_\varepsilon \to \delta v \ \text{weakly-* in } \mathcal{M}(\mathbf{R}) \ as \ \varepsilon \to 0,$$

where (δv) is the unique duality solution of the linear problem

$$\partial_x \left(f'(v)\delta v \right) + \delta v = \delta g \tag{4.17}$$

in the sense specified in Definition 4.1.

Remark 4.2. *In accordance with [3, Theorem 2.2], the function $f'(v)$ in (4.17) may be redefined on the set of zero measure in such a way that the resulting function is Borel and (4.17) holds in the sense of distributions. In particular, the singular part of the measure δv is supported by shocks of v.*

4.2. Sensitivity of shocks

As in the previous section, we assume that the flux function f is strictly convex with a (global) minimum attained in $\mathbf{R}$. Actually, to simplify the presentation we assume that f is even, as it is for instance the case for the Burgers equation where $f(u) = u^2/2$.

Assume that an entropy solution v exhibits a shock discontinuity at $x = 0 \in \operatorname{supp} g$. Our goal is to justify the following relation:

$$\delta\varphi = \lim_{\varepsilon \to 0} \frac{\varphi_\varepsilon}{\varepsilon} = \frac{[f'(v)\delta v]_0}{[v]_0} \tag{4.18}$$

where $[u]_{x_0} := u(x_0^+) - u(x_0^-)$ stands for the jump of a function u across $x = x_0$, φ_ε stands for the shock location of the perturbed solution $v_\varepsilon = v + \varepsilon\delta v_\varepsilon$ under forcing $g \equiv g + \varepsilon\delta g_\varepsilon$. In particular, we have

$$\lim_{x \to 0^-} v(x) = v(0^-) > v(0^+) = \lim_{x \to 0^+} v(x) = -v(0^-) \, . \tag{4.19}$$

Note that $v(0^+) = \lim_{x \to 0^+} v(x) = -v(0^-)$ since f is even and $f(v(0^+)) = f(v(0^-))$.

The function $x \mapsto f(v(x))$ is Lipschitz continuous (note that the Rankine-Hugoniot condition ensures in this case the continuity of $f(v)$ across the shock), in particular, there exist $a < 0$ and $b > 0$ such that the solution has the following structure

$$v(a) = 0, \ v(x) > 0 \text{ in } (a, 0], \ v \in C\left([a, 0]\right),$$

and

$$v(b) = 0, \ v(x) < 0 \text{ in } [0, b), \ v \in C\left([0, b]\right).$$

Accordingly, the solution may not have an infinite number of shocks accumulating at a point and shocks are isolated. Moreover, if g is as in Theorem 4.1, the function $f'(v)$ satisfies the one-sided Lipschitz condition (4.9) and the linearized equation

$$\partial_x(f'(v)\delta v) + \delta v = \delta g, \ \delta g \in L^1 \cap L^\infty,$$

admits a unique solution

$$\delta v \in L^1(a, 0) \text{ and } \delta v \in L^1(0, b).$$

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Consider now the solutions $v_\varepsilon = v + \varepsilon \delta v_\varepsilon$ of the perturbed problem

$$\partial_x \left(f(v_\varepsilon) \right) + v_\varepsilon = g + \varepsilon \delta g \,.$$

Since v_ε, v are entropy solutions, we get

$$||v_\varepsilon - v||_{L^1(\mathbf{R})} \leqslant \varepsilon \, ||\delta g||_{L^1(\mathbf{R})}$$

and, in particular,

$$\delta v_\varepsilon \to \delta v \text{ weakly-* in } \mathcal{M}(\mathbf{R}) \,.$$

In accordance with Theorem 4.1, the limit measure δv is unique, the so-called duality solution of the limiting stationary problem. We have

$$f(v_\varepsilon) \to f(v) \text{ in } C\left([a, b]\right),$$

and we deduce that v_ε possesses a unique shock discontinuity at the point $x = \varphi_\varepsilon$ in any compact interval $[a + \delta, b - \delta]$ for any $\delta > 0$ provided $\varepsilon = \varepsilon(\delta)$ is small enough.

Moreover, $\varphi_\varepsilon \to 0$ as $\varepsilon \to 0$ and one can quantify the variation of the shock location with respect to the perturbation $g \to g + \varepsilon \delta g$. Indeed, I_ε being an open interval with the end points 0, φ_ε. We have

$$||\delta v_\varepsilon - \delta v||_{C([a+\delta, b-\delta] \setminus I_\varepsilon)} \to 0 \text{ as } \varepsilon \to 0.$$

On the other hand,

$$\int_{I_\varepsilon} \delta v_\varepsilon \, dx = \frac{1}{\varepsilon} \int_{I_\varepsilon} (v_\varepsilon - v) \, dx = \frac{\varphi_\varepsilon}{\varepsilon} \frac{1}{|I_\varepsilon|} \int_\varepsilon |v_\varepsilon - v| \, dx,$$

where

$$\int_{I_\varepsilon} \delta v_\varepsilon \, dx \to [f'(v)\delta v] \, (0+) - [f'(v)\delta v] \, (0-)$$

while

$$\frac{1}{|I_\varepsilon|} \int_\varepsilon |v_\varepsilon - v| \, dx \to [v] \, (0-) - [v] \, (0+).$$

Thus we have proved (4.18).

4.3. Stationary problem seen as an evolutionary one

In this section, we still assume that the flux function f is strictly convex with a (global) minimum attained in $\mathbf{R}$.

The sensitivity analysis of the steady equation (1.1) with respect to forcing can be also obtained as a formal limit as $t \to \infty$ of the unsteady one (1.3) in view of the exponential stability of those solutions (see Corollary 2.1).

First of all we consider the evolution problem (1.3) and, for the sake of simplicity, we assume that the solution develops at most one shock. Let us assume that $u(t, x)$ is a Lipschitz continuous solution of equation (1.3) on $\Omega^\pm$ separated by a regular curve

$$\Sigma = \{(t, \varphi(t)), \, t > 0\},$$

where it satisfies the Rankine-Hugoniot condition

$$\varphi'(t) = \frac{[f(u)]_{\varphi(t)}}{[u]_{\varphi(t)}}$$

and the entropy condition $[u]_{\varphi(t)} \leqslant 0$.

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In the presence of shocks, the state of equation (1.3) needs to be viewed as a pair $(u(t, \cdot), \varphi(\cdot))$ and problem (1.3) can be written as follows:

$$
\begin{cases}
\partial_t u + \partial_x (f(u)) + u = g(x), \ t \geqslant 0, \ x \in \Omega(t), \\
\varphi'(t) = \dfrac{[f(u)]_{\varphi(t)}}{[u]_{\varphi(t)}}, \ t \in (0, T), \\
u(0, x) = u^0(x), \ x \in \Omega(0), \\
\varphi(0) = \varphi_0
\end{cases}
\tag{4.20}
$$

where $\Omega(t)$ stands for the union of the two regions to both sides of the shock:

$$\Omega(t) = \{x < \varphi(t)\} \cup \{x > \varphi(t)\}.$$

We then analyse the sensitivity of the pair (u, φ) with respect to perturbations δg of the source term g. The functional framework adopted here is based on the generalized tangent vectors introduced by Bressan and Marson [5] (see also [7]).

Let

$$g \in L^1 \cap L^\infty \cap BV(\mathbf{R}) \text{ with supp } g \text{ compact in } \mathbf{R}$$

be the source term. Let

$$u^0 \in L^1 \cap L^\infty(\mathbf{R}) \text{ with supp } u^0 \text{ compact in } \mathbf{R}$$

be the initial datum with a single discontinuity at

$$x = \varphi_0 \in \text{supp } g$$

and consider a generalized tangent vector

$$(\delta g, \delta u^0, \delta \varphi_0) \in (L^1 \cap L^\infty \cap BV(\mathbf{R})) \times (L^1 \cap L^\infty(\mathbf{R})) \times \mathbf{R}.$$

Then, for ε small enough, the perturbed solution $u_\varepsilon(t, \cdot) = u(t, \cdot) + \varepsilon \delta u(t, \cdot)$ of (1.3) with $g = g_\varepsilon (\equiv g + \varepsilon \delta g)$ is Lipschitz continuous developing a single discontinuity at $x = \varphi_\varepsilon(t)$ for $t > 0$.

Thus, $u_\varepsilon(t, \cdot)$ generates a generalized tangent vector

$$(\delta u(t, \cdot), \delta \varphi(t)) \in (L^1 \cap L^\infty(\mathbf{R})) \times \mathbf{R}.$$

One can prove (see for instance [4]) that it solves the following linearized system:

$$
\partial_t \delta u + \partial_x (f'(u) \delta u) + \delta u \;=\; \delta g, \quad (t, x) \in (0, T) \times \Omega(t), \tag{4.21}
$$

$$
\delta \varphi'(t) [u]_{\varphi(t)} + \delta \varphi(t) \left(\varphi'(t) [u_x]_{\varphi(t)} - [f'(u) u_x]_{\varphi(t)} \right) \tag{4.22}
$$

$$
+ \varphi'(t) [\delta u]_{\varphi(t)} - [f'(u) \delta u]_{\varphi(t)} \;=\; 0, \ t \in (0, T),
$$

$$
\delta u^0(x) \;=\; \delta u(0, x), \ x \in \Omega(0), \tag{4.23}
$$

$$
\delta \varphi(0) \;=\; \delta \varphi_0. \tag{4.24}
$$

This system has a unique solution which can be computed in two steps. Characteristics of (4.21) and (1.3) being the same, providing u solution of (1.3), the method of characteristics determines δu outside the shock curve. Thus the values of u and u_x to both sides of the shock curve allow to determine the jump relation through $x = \varphi(t)$. As a consequence all coefficients of the ODE are known and $\delta \varphi$ is then obtained by solving the equation (4.22).

In view of the linear structure of equation (4.21) with a possibly discontinuous coefficient $f'(u)$, the L^1 contraction property holds (see [1]). Thus, applying Corollary 2.1 to equations (4.21) and (4.25) with $u_0 = \delta u_0$, $u = \delta u$ and $v = \delta v$ yield to

$$\|\delta u(t, \cdot) - \delta v\|_{L^1(\mathbf{R})} \leq \exp(-t) \|\delta u^0 - \delta v\|_{L^1(\mathbf{R})} \text{ for any } t \geq 0.$$

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Thus, noting $\delta\varphi = \lim_{t\to+\infty}\delta\varphi(t)$, $\varphi = \lim_{t\to+\infty}\varphi(t)$, up to a subsequence, one can pass to the limit in all terms of equation (4.22). We get the following linearized steady equation

$$\partial_x(f'(v)\delta v) + \delta v = \delta g,\ x \in \mathbf{R}/\{\varphi\}, \tag{4.25}$$

$$\delta\varphi\,[f'(v)v_x]_\varphi + [f'(v)\delta v]_\varphi = 0. \tag{4.26}$$

Equation (4.26) provides the sensitivity of the shock location (which can be also obtained easily by linearizing the static shock condition $[f(v)]_\varphi = 0$ and already obtained, see equation (4.18)) and can be written as

$$\delta\varphi = \frac{[f'(v)\delta v]_\varphi}{[v]_\varphi}\ .$$

Let us illustrate the sensitivity of the shock location with respect to small perturbations of the source term for T large enough. We consider in Figure 1 $g_\varepsilon(x) = g(x) + \varepsilon\delta g(x)$ where

$$\delta g(x) = \begin{cases} 0 & \text{if} \quad 0 \leqslant x \leqslant \pi \\ g(x) & \text{if} \quad \pi < x \leqslant 2\pi \end{cases}$$

as a perturbation of $g(x) = \sin(x)$. In Figure 1(a) we show the behavior of the solution v_ε under the perturbation g_ε and in Figure 1(b) the sensitivity of the shock location. In particular, according to numerical simulations the shock seems to disappear for $\varepsilon \gtrsim 8.6$.

(a) v_ε
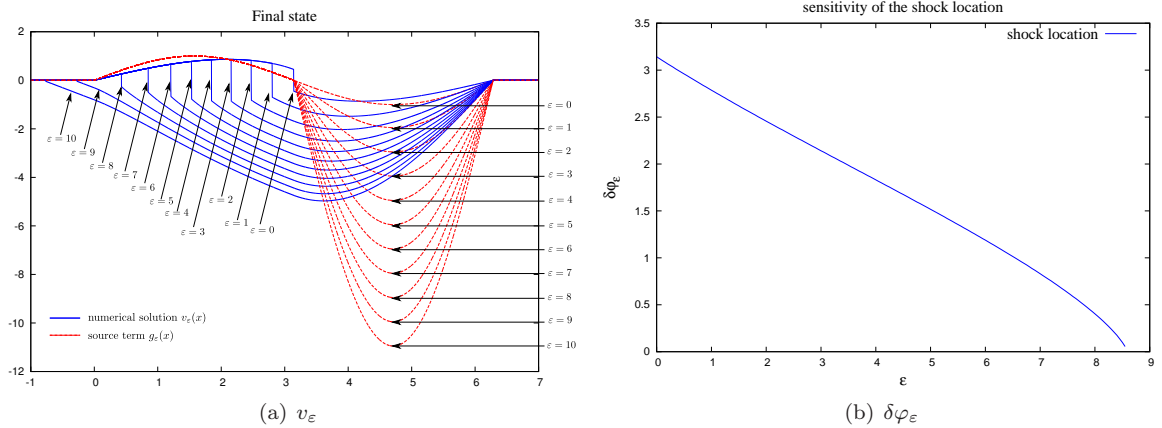
(b) $\delta\varphi_\varepsilon$

Figure 1: Numerical illustration of the sensitivity of $(v_\varepsilon, \delta\varphi_\varepsilon)$

5. Examples, extensions, concluding remarks

In this section, throughout several examples, we describe some qualitative properties of steady solutions such as their support, shock location, *etc.* We also discuss a property that numerical simulations seem to indicate, according to which, solutions, at least in part, stabilize in finite time. This constitutes an interesting open problem.

For the sake of simplicity, let us again consider the Burgers equation (1.1) with $f(v) = v^2/2$ and $g \in L^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$ of compact support.

5.1. Structure of steady solutions

In what follows, given a finite number of points $\{a_0, \ldots, a_N\}$ we consider g as the sum of

$$g(x) = \sum_{i=0}^{N-1} g_i(x)\mathbb{1}_{(a_i, a_{i+1})}(x) \tag{5.1}$$

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where $g_i(x) = g(x)_{|(a_i, a_{i+1})}$ is either positive or negative and so that sign changes from a subinterval to the other.

Following Mascia and Terracina [13], under such circumstances, one can easily prove the following:

Properties 5.1. *For a given $1 \leqslant i \leqslant N$, the stationary solution has at most one shock $s \in [a_{i-1}, a_i]$.*

Proof. Assume by contradiction that there exist two shocks, say, $(s_1, s_2) \in [a_{i-1}, a_i]^2$ with $s_1 < s_2$. Then there exist three C^1 solutions of problem (1.1) v^-, v^m and v^+ such that the solution on the whole interval $[a_{i-1}, a_i]$ is:

$$v(x) = v^-(x)\mathbb{1}_{(a_{i-1} \leqslant x < s_1)}(x) + v_m(x)\mathbb{1}_{(s_1 < x < s_2)}(x) + v^+(x)\mathbb{1}_{(s_2 < x \leqslant a_i)}(x)$$

where

$$v^-(s_1^-) > 0 > v_m(s_1^+) \text{ and } v_m(s_2^-) > 0 > v^+(s_2^+).$$

On one hand, since we have $v_m(s_1^+) < 0 < v_m(s_2^-)$ then there exists

$$s_1 < s < s_2 \text{ such that } v_m(s) = 0.$$

On the other, $v_m(x)$ satisfies

$$\forall x \in (s_1, s_2), f'(v_m(x))v_m'(x) = g(x) - v_m(x)$$

with

$$f'(v_m(s)) = f'(0) = 0.$$

As a consequence, we have $g(s) = 0$ which is a contradiction with the decomposition (5.1) of g. $\square$

In what follows, we illustrate examples of steady solutions of equation (1.1) for $f(v) = v^2/2$ when considering continuous source terms (see Figure 2(a) and Figures 3-4),

$$g_c(x) = \sin(4x)\mathbb{1}_{(0,2\pi)}(x)$$

and discontinuous source terms (see Figure 2(b)),

$$g_{dc}(x) = \mathbb{1}_{(0,1)}(x) - 0.8\,\mathbb{1}_{(1,2)}(x) + 0.4\,\mathbb{1}_{(2,3)}(x) - 0.5\,\mathbb{1}_{(3,4)}(x) + 0.7\,\mathbb{1}_{(4,5)}(x)$$

$$-2\,\mathbb{1}_{(5,6)}(x) + 2\,\mathbb{1}_{(6,7)}(x) - \mathbb{1}_{(7,8)}(x) + 3\,\mathbb{1}_{(8,9)}(x).$$

The results are displayed on Figure 2 and, for both cases, there is only one shock per interval where g keeps the same sign.
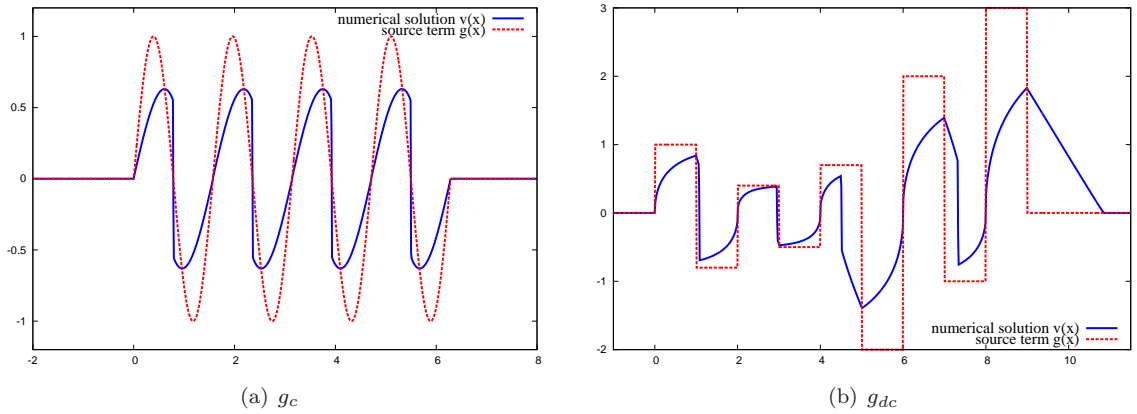

Figure 2: Multiple steady shocks

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Note also that, while in the first example, the support of the steady solution v is included in the one of the forcing term g_c, that is not the case in the second case. Indeed, in that one the support of v goes beyond that of g_{dc} to the right. Over there the solution exhibits a simple linear behavior in agreement with the equation fulfilled beyond the support of the forcing term $(\partial_x(v^2/2) + v = 0)$.

5.2. Evolution problem

Let us come back to the two previous examples. The inspection of the time evolution of several numerical simulations [4] seems to indicate that there exists one part of the solution (the one localized on the support of g) that stabilizes in finite time (this can be observed, for instance in Figure 3), while the other one decays exponentially to zero (as displayed on Figure 4) beyond the support of g where the equation reads $\partial_t u + \partial_x f(u) + u = 0$.

More precisely, it seems that the following assertion holds:

Conjecture: *For any u^0 and g with compact support there exists a finite time $T^* > 0$ such that*

$$\forall t \geqslant T^*, \ \forall x \in supp \ g, \quad u(t, x) = v(x),$$

where v is the unique entropy steady state solution.

---

[4]We have used the standard Godunov scheme to compute each numerical solution on a fine grid with $N = 10000$ with free boundary conditions.
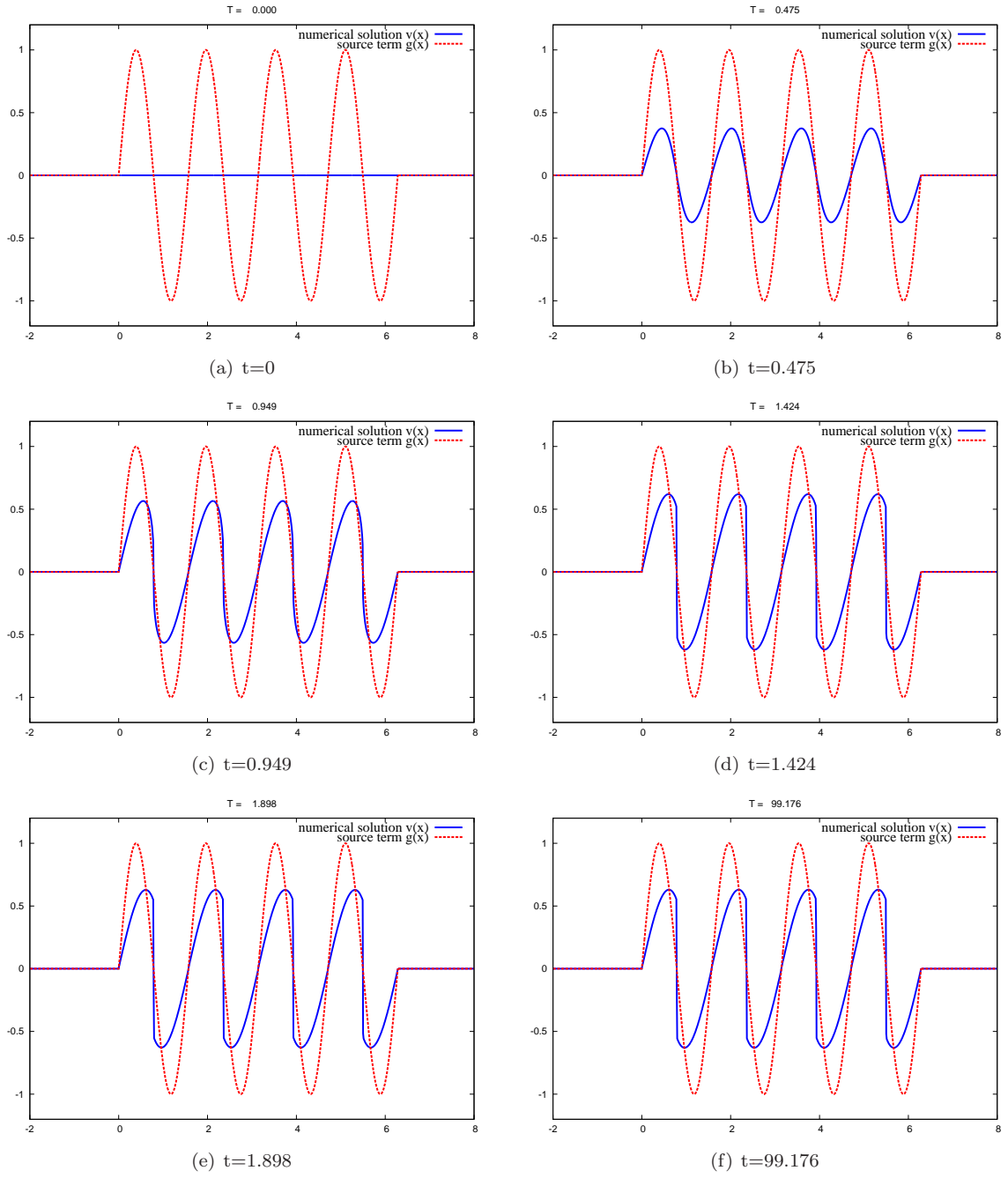
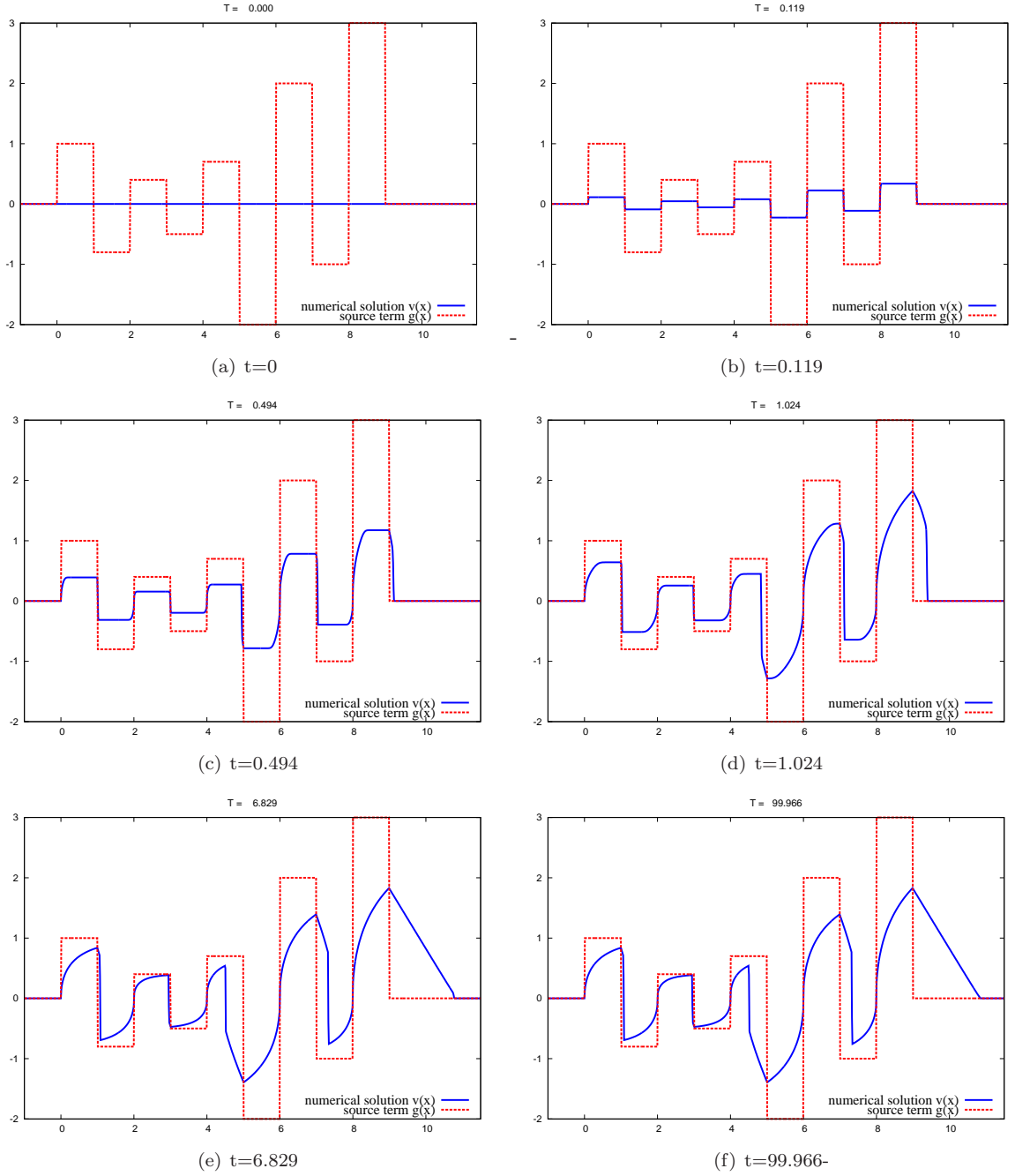Figure 3: A steady solution emerging in finite time with g_c

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Figure 4: A local finite time emerging steady solution with g_{dc}

Note that rigorous analysis leads to the exponential convergence of the total mass but that the behavior inside the support of the forcing term g requires further work. Indeed, let $m(t) = \int_{\mathbf{R}} u(t,x)\,dx$ be the mass of u over $\mathbf{R}$. It satisfies

$$m' + m = \int_{\mathbf{R}} g\,dx.$$

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Thus, one has

$$
\begin{array}{rcl}
m(t) & = & \int_{\mathbf{R}} g(x)\, dx + \left(m(0) - \int_{\mathbf{R}} g(x)\, dx\right) e^{-t} \\
& = & \int_{\mathbf{R}} v(x)\, dx + \left(m(0) - \int_{\mathbf{R}} g(x)\, dx\right) e^{-t}
\end{array}
$$

since $\int_{\mathbf{R}} v(x)\, dx = \int_{\mathbf{R}} g(x)\, dx$. For a better understanding of the large time behavior and the possible stabilization in finite time a finer analysis of the characteristic of equation (1.3) is required.

The numerical simulations seem to indicate the following:

- In Figure 3, one has $m(0) = \int_{\mathbf{R}} g(x)\, dx = 0$ and thus $m(t) = 0$, for all $t > 0$. Moreover it seems that supp $u \subset$ supp g, for all $t > 0$. Such a solution should stabilize in finite time.

- In Figure 4, one has $m(0) = 0$, $\int_{\mathbf{R}} g(x)\, dx \neq 0$ and thus $m(t) = (1 - e^{-t}) \int_{\mathbf{R}} g(x)\, dx$, for all $t > 0$. Part of $u(t)$ escapes supp g, then it seems to converge to zero exponentially but not in finite time. It seems that, nevertheless, the part of the solution restricted to supp g reaches the steady state in finite time.

  As we said above these global or local finite time stabilization properties constitute interesting open problems to be rigorously analyzed.

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