High-gain extended Kalman filter for continuous-discrete systems with asynchronous measurements

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Abstract

This paper investigates an adaptation of the high-gain Kalman filter for nonlinear continuous-discrete system with multirate sampled outputs under an observability normal form. The contribution of this article is twofold. First, we prove the global exponential convergence of this observer through the existence of bounds for the Riccati matrix. Second, we show that, under certain conditions on the sampling procedure, the observer's asynchronous continuous-discrete Riccati equation is stable and also, that its solution is bounded from above and below. An example, inspired by mobile robotics, with three outputs available is given for illustration purposes.

1 Introduction

The present paper deals with the design of observers for nonlinear multirate sampled-data systems under asynchronous sampling —i.e. control systems having continuous state dynamics and a discrete measurement procedure. This situation arises when the output vector of a control system is obtained through several sensors that do not have the same (possibly non-uniform) sampling rate. Such systems are often met in practice, for instance in global positioning problems, as in [33], or in the field of drone control [10]. Likewise, one can be confronted with such asynchronous systems in the fields of submarine robotics, as can be seen from [3, 31, 6], chemical engineering [41], or cultivation engineering [4].

As emphasised in [43], this state estimation problem can be tackled by considering one of the three following options. First, model the state dynamics as discrete and apply a known estimator for discrete state systems —see for example [4] in the linear setting. Second, lift the measurements into the space of continuous functions, e.g. with the help of a polynomial fit as in [41] in the nonlinear setting. Third, directly consider the continuous model for the state dynamics and the discrete model for the measurements. This latter option is

the one retained in the present paper, for nonlinear systems, in the framework of high-gain observers [18].

Considering the design of observers, or estimators, for linear multirate stochastic systems, [43] posed the problem in terms of Itô-Volterra equations associated to discrete measurements, which allowed them to derive a very general optimal filter in this framework. Using the theory of vibrosolutions of integral equations with discontinuous measures, the authors provided an explicit solution in the form of a Kalman-like estimator. More recently, in [34], the authors modeled each sensor as a sample-and-hold device and performed a stability analysis based on Lyapunov-Krasovskii functionals. They also considered the problem of determining the maximum time interval between consecutive measurements that guarantees exponential stability. It was addressed under the guise of an optimisation problem in terms of linear matrix inequalities (LMI). In [28], the authors built upon the ideas of [25, 1] where an already designed continuous-time Luenberger-like observer was coupled with asynchronous inter-samples predictors. Finally, the problem under consideration was also addressed by using multirate versions of the Kalman filter, see for instance [27, 4, 24, 17]. In particular, in [17], the authors studied the exponential convergence of the proposed observer and the preservation of observability for multirate systems. The present article extends this latter approach to nonlinear systems within the framework of high-gain observers.

In the nonlinear framework, there are many paths one can follow in order to perform data fusion for multirate systems, as it can be seen from [22]. In [41], the authors relied on a fully continuous Luenberger-type design where the missing measurements were predicted with the help of a polynomial interpolation method. More recently, [29] used an already designed continuous-time, Luenberger-like, observer coupled with asynchronous inter-samples predictors. Also relying on a fixed correction gain, [40] proposed a continuous observer for multirate systems where the measurements were updated whenever available, the sensors being seen as sample-and-hold devices. In this latter paper, the global exponential stability of the observer was proved assuming that the system under consideration is under an observability normal form distinct from the one used in the present work —see [19] for details.

Let us mention two more contributions based on Luenberger-like designs. In [42], the authors addressed the problem of robust multirate estimation in the sense that measurements were available in two time scales: *fast* and *slow*. There, the *slow* measurements were shown to enhance the robustness of the estimation procedure with respect to modelling errors. For this purpose, the state variables need to be (locally) *integral detectable* from the *slow* measurements. Finally, in [11], the authors proposed a discrete-time state estimation based on the Taylor series expansion of the system's dynamics. The analysis of the proposed observer follows the ideas of [18] regarding systems that are *observable for any inputs* but without using an explicit high-gain parameter.

A multirate moving horizon estimator was detailed in [30]. It relied on a binary switching sequence in order to model the multirate sampling and predictions of the missing measurements.

Finally, the extended Kalman filter design has also been considered for multirate estimation, as it can be seen from [13, 14, 21, 35] where systems having two time scales were considered. In [10] a multirate extended Kalman filter was considered to perform data fusion onboard a small-scale helicopter.

The present paper details the design of a high-gain extended Kalman filter for the state estimation of multirate nonlinear systems. Following the ideas of [23, 12, 8, 17] the proposed observer consists of two steps: (i) an open-loop prediction when no measurements are available, and (ii) an impulsive correction each time a new measurement is available. This second step is performed according to the actually measured outputs which may consist of a subset of the system's output vector only. The global exponential convergence is proven under the hypothesis that the system is under an observability normal form —see e.g. [19, 2, 15]. The main difficulties are, on the one hand, to deal with several non-uniform subdivisions of time in order to represent the asynchronous outputs, and on the other hand, proving that the observer's Riccati equation is bounded over time. This latter issue is handled by following the ideas developed in [7], where only the synchronous setting is considered.

In the present paper, as usual with the high-gain framework, the convergence analysis is done completely in the noise-free setup. However, high-gain estimators are known for being very sensitive to noise. Since our design is based on an extended Kalman filter, which has good noise filtering properties [36], we expect the filtering efficiency of the Kalman design to counterweight the drawbacks of the high-gain formalism —as it is the case in our example in Section 5. As this might not be sufficient, a natural extension to the present work is to consider a varying high-gain parameter in order to get the best of both worlds, as it was done in [9, 16, 38].

The remainder of the article is as follows. In Section 2, the system under consideration is introduced. In particular, it introduces the notion of *virtual sensor* in order to take into account measurements that are always available at the same time steps. The observer proposed for this class of systems is defined in Section 3. Section 4 deals with the proof of the global exponential convergence of this observer. The demonstration heavily relies on the existence of bounds for the solution to the observer's Riccati equation. For the sake of clarity in the exposure, the proof of this result is given in appendix A. It basically follows the ideas developed in [7], with an increased complexity coming from the asynchronicity of the measurements that makes this exposure necessary. Section 5 is dedicated to an example coming from mobile robotics. Finally, Section 6 concludes the article.

Notations

- A **time subdivision** $\{\tau_k\}_{k\in\mathbb{N}}$ is meant as a strictly increasing sequence of real numbers with $\tau_0 = 0$ and $\tau_k \to \infty$ when $k \to \infty$.

- Id is the identity matrix with appropriate dimensions, $diag[v]$ denotes a diagonal matrix whose elements are the elements of v. Throughout the

paper, v can either be a vector or a set of matrices. In this latter case, $diag[v]$ is to be understood as a block-diagonal matrix.

- For a square matrix M, $Tr(M)$ denotes the trace.
- $\mathbb{R}^*$ means $\mathbb{R}\backslash\{0\}$, and $\mathbb{N}^*$ stands for $\mathbb{N}\backslash\{0\}$.
- If Ω is a set, we denote by $|\Omega|$ the cardinality of this set.
- *w.r.t.* is used as the short form of *with respect to*, and *s.p.d.* stands for *symmetric positive definite*. Oftentimes, time dependencies are omitted to make the notation less cluttered.
- For a time varying quantity $x(\tau)$ evaluated at time τ_k, we use the notation $x_k = x(\tau_k)$. At times τ_k, correction steps are performed. At such times, the quantity $x(\tau_k)$ is denoted $x_k^{(-)}$ before the correction step, and $x_k^{(+)}$ after.
- Let us consider a product of time varying quantities (having appropriate dimensions) of the form: $x^{'}(\tau)S(\tau)x(\tau)$. Then, at time τ_k, the notation $\left(x^{'}Sx\right)_k^{(-)}$ is the short form of $x_k^{(-)'}S_k^{(-)}x_k^{(-)}$.

2 System under consideration

Let (Σ_c) be a nonlinear, observable, continuous system under the following observability normal form —see also [16, 19, 39]:

$$\begin{cases} \dot{x}(\tau) = & A(u(\tau))x(\tau) + b(x(\tau), u(\tau)) \quad \text{with} \quad x(0) = x_0 \\ y(\tau) = & C(\tau)x(\tau) \end{cases} \tag{Σ_c}$$

The state variable $x(\tau)$ lies in a compact subset χ of $\mathbb{R}^n$, the output $y(\tau)$ is in $\mathbb{R}^{n_y}$ and the input vector $u(\tau)$, which belongs to $\mathcal{U}_{adm} \subset R^{n_u}$, is bounded for all times. The state variable is decomposed into n_y subvectors as follows:

$$x(\tau) = \begin{pmatrix} x_1(\tau) \\ \vdots \\ x_{n_y}(\tau) \end{pmatrix}$$

where, for all $i \in \{1, ..., n_y\}$, $x_i(\tau)$ is in the compact subset $\chi_i \subset \mathbb{R}^{n_i}$ (and $\sum_{i=1}^{n_y} n_i = n$). Each subvector $x_i(\tau)$ is associated to the ith output $y^i(\tau)$ and is written

$$x_i(\tau) = \begin{pmatrix} x_i^1(\tau) \\ \vdots \\ x_i^{n_i}(\tau) \end{pmatrix}$$

The dynamics of $x_i(\tau)$ are described by:

$$\begin{cases} \dot{x}_i(\tau) & = & A_i(u(\tau))x_i(\tau) + b_i(x(\tau), u(\tau)) \\ y^i(\tau) & = & C^i(u(\tau))x_i(\tau) \end{cases}$$

- $A_i(u)$ and $C^i(u)$ are, respectively, (n_i, n_i) and $(1 \times n_i)$ matrices of the form

$$A_i(u) = \begin{pmatrix} 0 & a_i^2(u) & \cdots & 0 \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & a_i^{n_i}(u) \\ 0 & \dots & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad C^i(u) = \begin{pmatrix} a_i^1(u) & 0 & \cdots & 0 \end{pmatrix}$$

  where, for all $i \in \{1, ..., n_y\}$, $j \in \{1, ..., n_i\}$, $u \in \mathcal{U}_{adm}$, $0 < a_m < |a_i^j(u)| < a_M$. Moreover, we suppose that the elements of the C^i matrices are differentiable at least once and have their derivative bounded over time[1].

- $b_i(x, u)$ is a $\mathcal{C}^1$ triangular, compactly supported, vector field whose last component is allowed to depend on the full state of (Σ_c):

$$b_i(x, u) = \begin{pmatrix} b_i^1(x_i^1, u) \\ b_i^2(x_i^1, x_i^2, u) \\ \vdots \\ b_i^{n_i-1}(x_i^1, ..., x_i^{n_i-1}, u) \\ b_i^{n_i}(x, u) \end{pmatrix}$$

  We assume that the Jacobian matrix $D_x b(x, u)$ of $b(x, u)$, computed *w.r.t.* x, is bounded from above by $L_b > 0$. Therefore, $b(x, u)$ has the Lipschitz property *w.r.t.* x (uniformly *w.r.t.* u): $\|b(x, u) - b(z, u)\| \leq L_b \|x - z\|$.

Finally, the full dynamics of system (Σ_c) are given by

$$A(u) = diag\left[A_1, \dots, A_{n_y}\right], \quad b(x, u) = \begin{pmatrix} b_1(x, u) \\ \vdots \\ b_{n_y}(x, u) \end{pmatrix} \text{ and } C(u) = diag\left[C^1, \dots, C^{n_y}\right]$$

To this plant, we associate the following continuous-discrete system with asynchronous, or multirate, measurements

$$\begin{cases} \dot{x}(\tau) = & A(u(\tau))x(\tau) + b(x(\tau), u(\tau)) \quad \text{with} \quad x(0) = x_0 \\ y(\tau_k) = & C_{\sigma_k} x(\tau_k) \end{cases} \qquad (\Sigma_{acd})$$

Contrary to what was proposed in a previous work [17], the asynchronous measurement procedure is not modelled with respect to each output. Instead, we lump together outputs always available at the same time.

[1] Although restrictive, this condition is necessary in order to apply Lemma 13 to a time varying matrix C.

1. Let a **sensor** be a non-empty subset $\mathfrak{s}_i \subset \{1, \dots, n_y\}$. It is associated to a vector $y^{(\mathfrak{s}_i)}(\tau) = \left\{y^j(\tau) \ : \ j \in \mathfrak{s}_i\right\}$. There are n_s sensors, with $0 < n_s \leq n_y$. In this work, we assume that the set of all sensors of (Σ_{acd}) is a partition of the set $\{1, \dots, n_y\}$. Henceforth, it is assumed that a sensor is made of consecutive indices. Indeed, this can always be achieved via a simple re-ordering of the output and state variables.

   The $(|\mathfrak{s}_i| \times n)$ output matrix corresponding to a sensor $\mathfrak{s}_i$ is denoted $C^{(\mathfrak{s}_i)}$ and is such that $y^{(\mathfrak{s}_i)}(\tau) = C^{(\mathfrak{s}_i)} x(\tau)$.

2. A subdivision of time $\left\{s_k^{(i)}\right\}_{k \in \mathbb{N}}$ is associated to each sensor $\mathfrak{s}_i$, and the global time subdivision $\{\tau_k\}_{k \in \mathbb{N}}$ is defined as:

$$\{\tau_k\}_{k \in \mathbb{N}} := \bigcup_{i=1}^{n_s} \left\{s_l^{(i)}\right\}_{l \in \mathbb{N}}$$

   where elements belonging to several subdivisions $\left\{s_l^{(i)}\right\}$ are considered only once.

3. For each τ_k there exist at least one sensor $\mathfrak{s}_i$ such that $s_l^{(i)} = \tau_k$ for some index l. Let σ_k denote the set of such sensors:

$$\sigma_k = \left\{i \in \{1, ..., n_s\} | \exists \ l \in \mathbb{N} \text{ such that } s_l^{(i)} = \tau_k\right\}$$

   The above mentioned l index, probably differs from k, and is not the same from sensor to sensor. As such, for all $i \in \sigma_k$, $l_k^{(i)}$ denotes the index $l \in \mathbb{N}$ such that $s_l^{(i)} = \tau_k$.

   The matrix C_{σ_k} associated to the set σ_k is the $\left(\sum_{i \in \sigma_k} |\mathfrak{s}_i| \times n\right)$ matrix made of the $C^{(\mathfrak{s}_i)}$ matrices that correspond to the output actually available at time τ_k:

$$C_{\sigma_k} = \begin{pmatrix} \vdots \\ C^{(\mathfrak{s}_i)} \\ \vdots \end{pmatrix}_{i \in \sigma_k} \qquad \text{and thus} \quad y_k = y(\tau_k) = C_{\sigma_k} x(\tau_k)$$

3 Definition of the multirate high-gain Kalman filter

The continuous-discrete asynchronous high-gain Kalman filter is defined in two parts:

1. two prediction equations when $\tau \in [\tau_{k-1}, \tau_k[$, $k \in \mathbb{N}^*$, with initial values $z_{k-1}^{(+)}$ and $S_{k-1}^{(+)}$;

2. two correction equations at time τ_k.

Notations:

- $z(\tau)$ is the estimated state for all $\tau \in [\tau_{k-1}, \tau_k[$;
- $z_k^{(-)}$ is the estimated state at time τ_k, at the end of a prediction step and before a correction step;
- $z_k^{(+)}$ is the estimated state at time τ_k, after a correction step. Therefore, $z_k^{(+)}$ is the initial estimated state of the new prediction interval $[\tau_k, \tau_{k+1}[$.

Prediction equations

$$\begin{cases} \dot{z}(\tau) = & A(u)z(\tau) + b(z,u) \\ \dot{S}(\tau) = & -\left(A(u) + D_x b(z,u)\right)' S(\tau) - S(\tau)\left(A(u) + D_x b(z,u)\right) - (SQ_\theta S)(\tau) \end{cases} \tag{$\mathcal{O}_1$}$$

Correction equations

$$\begin{cases} z_k^{(+)} = & z_k^{(-)} - S_k^{(+)^{-1}} \sum_{i \in \sigma_k} C^{(\mathfrak{s}_i)'} \left(R_\theta^{(\mathfrak{s}_i)}\right)^{-1} \left(C^{(\mathfrak{s}_i)} z_k^{(-)} - y_k^{(\mathfrak{s}_i)}\right) \left(s^{(i)}_{l_k^{(i)}} - s^{(i)}_{l_k^{(i)}-1}\right) \\ S_k^{(+)} = & S_k^{(-)} + \sum_{i \in \sigma_k} C^{(\mathfrak{s}_i)'} \left(R_\theta^{(\mathfrak{s}_i)}\right)^{-1} C^{(\mathfrak{s}_i)} \left(s^{(i)}_{l_k^{(i)}} - s^{(i)}_{l_k^{(i)}-1}\right) \end{cases} \tag{$\mathcal{O}_2$}$$

In other words, the correction at a time τ_k is made with respect to each measurement $y_k^{(\mathfrak{s}_i)}$ that is actually available and involves a weighting factor equal to the time elapsed since the last time this specific output was measured.

The matrices Q_θ and $R_\theta^{(\mathfrak{s}_i)}$ —which can be time dependent[2] provided the constraints (1)-(2) below are met— are of the form

$$Q_\theta = \theta \Delta^{-1} Q \Delta^{-1} \quad \text{and} \quad R_\theta^{(\mathfrak{s}_i)} = \frac{1}{\theta} \delta^{(\mathfrak{s}_i)} R^{(\mathfrak{s}_i)} \delta^{(\mathfrak{s}_i)}$$

where

- Q and $R^{(\mathfrak{s}_i)}$ are *s.p.d.* matrices, of dimensions $(n \times n)$ and $(|\mathfrak{s}_i| \times |\mathfrak{s}_i|)$ respectively which must lie in compact subsets such that:

$$\underline{q}\ \mathrm{Id} \ \leq Q \leq \overline{q}\ \mathrm{Id} \quad \text{with} \quad 0 < \underline{q} < \overline{q} \tag{1}$$

$$\underline{r}_i\ \mathrm{Id} \ \leq R^{(\mathfrak{s}_i)} \leq \overline{r}_i\ \mathrm{Id} \quad \text{with} \quad 0 < \underline{r}_i < \overline{r}_i \tag{2}$$

- $\delta^{(\mathfrak{s}_i)}$ and Δ are both diagonal matrices whose construction relies on the quantity $n^* = \max(n_1, n_2, \ldots, n_{n_y})$ and on a fixed scalar $\theta \geq 1$:

$$\Delta = diag\left[\Delta_1, \ldots, \Delta_{n_y}\right] \quad \text{where} \quad \Delta_i = diag\left[\frac{1}{\theta^{n^*-n_i}}, \ldots, \frac{1}{\theta^{n^*-1}}\right]$$

$$\text{and} \quad \delta^{(\mathfrak{s}_i)} = diag\left[\left\{\theta^{n^*-n_j} \ : \ j \in (\mathfrak{s}_i)\right\}\right]$$

[2]This time dependency is not explicitly written in the observer's equations to make the notations less cluttered.

Finally, $R = diag\left[R^{(\mathfrak{s}_1)}, \ldots, R^{(\mathfrak{s}_{n_s})}\right]$ and $R_\theta = diag\left[R_\theta^{(\mathfrak{s}_1)}, \ldots, R_\theta^{(\mathfrak{s}_{n_s})}\right]$, or equivalently, if one defines $\delta = diag\left[\delta^{(\mathfrak{s}_1)}, \ldots, \delta^{(\mathfrak{s}_{n_s})}\right]$: $R_\theta = \frac{1}{\theta}\delta R \delta$.

The initial datum of the observer is made of the initial estimated state $z(0) \in \chi \subset \mathbb{R}^n$ and of $S(0)$, a *s.p.d.* matrix.

Remark 1. *1. When θ equals 1, the proposed observer can be seen as the nonlinear version of the continuous-discrete Kalman filter. In this context, the time ponderation in ($\mathcal{O}_2$) is unusual. However, this time ponderation is critical in the proof of convergence. In fact, ($\mathcal{O}_2$) corresponds to the discretisation of a continuous extended Kalman filter, but performed with respect to the asynchronicity of the measurements. As such, the scaling factor is necessary so that equations keep a mathematical meaning when the discretisation step goes to zero.*

From a practical perspective, this time ponderation prevents from giving a too high weight to high-frequency measurements compared to low-frequency measurements.

2. The two matrices Q_θ and R_θ, built according to the normal form of an observable system, constitute the high-gain formalism. The fixed parameter θ is the so-called high-gain parameter. When $\theta = 1$, the proposed observer is a simple extended Kalman filter for which the normal form allows to prove local convergence only —see e.g. [9].

Although out of scope of the present work, a worth mentioning issue is the study of methods that allow to define and run the observer in the original coordinates of the system instead of the normal coordinates. Interested readers can refer to, e.g., [5, 26, 37] and references herein.

3. Although the definitions of Δ and δ may appear uselessly intricate, they are necessary in order to simplify forthcoming computations, in particular by preserving the Lipschitz constant of vector field $b(x, u)$ despite the change of variables performed at the beginning of the proof of convergence (cf. Sec. 4).

4. According to Equation ($\mathcal{O}_2$), R cannot be any s.p.d. matrix. It is in fact a block diagonal s.p.d. matrix, each block corresponding to a sensor. In [17], R was a diagonal matrix and the approach less general. Here, our definition of sensors allows us to consider correlations between potential measurement noises that corrupt measurements given by the same physical sensor.

5. In the framework of high-gain observers, matrices Q and R are viewed as tuning parameters because of the influence of θ and the noise-free setup. However, in practice, these matrices can be chosen based on the noise characteristics in order to reflect differences in scales or correlations between potential measurement noises, see for example [39].

4 Convergence of the asynchronous high-gain filter

Theorem 1. *Consider* (Σ_{acd})*, a continuous-discrete system with asynchronous measurements obtained from an observable nonlinear system* (Σ_c) *under the assumptions presented in Section 2.*

Let $x(\tau) \in \mathbb{R}^n$ *and* $y(\tau) \in \mathbb{R}^{n_y}$ *be the state and output vectors of* (Σ_{acd})*. Let* $n_s \geq 1$ *be an integer such that* $y(\tau)$ *is made of* n_s *sub-vectors. For a given* $i \in \{1, ..., n_s\}$*, the dimension of the corresponding sub-vector of* $y(\tau)$ *is denoted* n_i *and a, possibly non-uniform, subdivision of time denoted* $\left\{s_k^{(i)}\right\}_{k\in\mathbb{N}}$ *tells when a measurement is actually available. For each* i*, let* $\delta t^{(i)}$ *denote the maximum step size of* $\left\{s_k^{(i)}\right\}_{k\in\mathbb{N}}$*.*

The asynchronous high-gain extended Kalman filter based on (Σ_{acd}) *is given by* ($\mathcal{O}_1$-$\mathcal{O}_2$)*, whose state is denoted by* $z(\tau) \in \mathbb{R}^n$*.*

Then, for any fixed $T > 0$*, there are positive constants* K_1*,* K_2*,* K_3*,* $\theta_0 > 1$*, and* μ_i*, for* $i = 1, ..., n_s$*, such that for all* $\theta > \theta_0$ *and* $\theta\delta t^{(i)} < \mu_i$*, one has for all* $\tau \geq \frac{T}{\theta}$*:*

$$\|z(\tau) - x(\tau)\|^2 \leq \left\| z\left(\frac{T}{\theta}\right) - x\left(\frac{T}{\theta}\right)\right\|^2 \theta^{2(n^*-1)} K_1 e^{(K_2-\theta K_3)(\tau-\frac{T}{\theta})} \tag{3}$$

where $n^* = \max\limits_{i\in\{1,...,n_s\}} \{n_i\}$*.*

Remark 2.

1. *As it can be seen from* (3)*,* θ *must be high enough to ensure the negativity of* $K_2 - \theta K_3$*.*
2. *The exact expressions of* K_1*,* K_2 *and* K_3 *appear at the end of the proof, in* (11)*. Also, those quantities depend on* T*.*

The proof of this theorem relies on the analysis of the dynamics of the estimation error: $\varepsilon(\tau) = z(\tau) - x(\tau)$. In the following, it is divided into two parts: the *preparation for the proof*, and the *exponential convergence*.

Preparation for the proof

Let us first consider the change of variables $\tilde{x} = \Delta x$, $\tilde{z} = \Delta z$ and $\tilde{\varepsilon} = \Delta\varepsilon$. We also denote $\tilde{b}(., u) = \Delta b(\Delta^{-1}., u)$, $D_{\tilde{x}}\tilde{b}(., u) = \Delta D_x b(\Delta^{-1}., u)\Delta^{-1}$ and $\tilde{S} = \Delta^{-1} S \Delta^{-1}$.

Lemma 2. [19]

1. *The vector field* $\tilde{b}(\tilde{x}, u)$ *has the same Lipschitz constant as* $b(x, u)$*.*
2. *The Jacobian* $D_x\tilde{b}(\tilde{x}, u)$ *has the same bound as* $D_x b(x, u)$*.*
3. *We also have the following relations:*

- $\Delta A = \theta A \Delta$, *and* $A\Delta^{-1} = \theta \Delta^{-1} A$;
- $\delta^{(\mathfrak{s}_i)^{-1}} C^{(\mathfrak{s}_i)} \Delta^{-1} = C^{(\mathfrak{s}_i)}$;
- $\Delta^{-1} C^{(\mathfrak{s}_i)'} R_\theta^{(\mathfrak{s}_i)^{-1}} C^{(\mathfrak{s}_i)'} \Delta^{-1} = \theta C^{(\mathfrak{s}_i)'} R^{(\mathfrak{s}_i)^{-1}} C^{(\mathfrak{s}_i)}$.

This change of variables allows us to remove the θ-dependency of the matrices $R_\theta^{(\mathfrak{s}_i)}$ and Q_θ. With the help of the relations given in Lemma 2, the observer's equations ($\mathcal{O}_1$)-($\mathcal{O}_2$) become:

$$\begin{cases} \dot{\tilde{z}}(\tau) = & \theta A(u)\tilde{z}(\tau) + \tilde{b}(\tilde{z}, u) \\ \dot{\tilde{S}}(\tau) = & -\left(\theta A(u) + D_{\tilde{x}}\tilde{b}(\tilde{z}, u)\right)' \tilde{S} - \tilde{S}\left(\theta A(u) + D_{\tilde{x}}\tilde{b}(\tilde{z}, u)\right) - \theta \tilde{S} Q \tilde{S} \end{cases} \tag{$\tilde{\mathcal{O}}_1$}$$

$$\begin{cases} \tilde{z}_k^{(+)} = & \tilde{z}_k^{(-)} \\ & -\theta \tilde{S}_k^{(+)^{-1}} \sum_{i \in \sigma_k} C^{(\mathfrak{s}_i)'} R^{(\mathfrak{s}_i)^{-1}} C^{(\mathfrak{s}_i)} \left(\tilde{z}_k^{(-)} - \tilde{x}(\tau_k)\right) \left(s_{l_k^{(i)}}^{(i)} - s_{l_k^{(i)}-1}^{(i)}\right) \\ \tilde{S}_k^{(+)} = & \tilde{S}_k^{(-)} + \theta \sum_{i \in \sigma_k} C^{(\mathfrak{s}_i)'} R^{(\mathfrak{s}_i)^{-1}} C^{(\mathfrak{s}_i)} \left(s_{l_k^{(i)}}^{(i)} - s_{l_k^{(i)}-1}^{(i)}\right) \end{cases} \tag{$\tilde{\mathcal{O}}_2$}$$

In order to proceed with the proof, we want to be able to bound all the elements of $\theta A(u) + D_{\tilde{x}}\tilde{b}(\tilde{z}, u)$, independently from θ. This is true for the lower bound since $\theta \geq 1$, but not for the upper bound. This issue is resolved with the help of a time reparametrization.

Let $\bar{\tau}$ be such that $\bar{\tau} = \theta\tau$. This leads to a change on the subdivisions $\{\tau_k\}_{k\in\mathbb{N}}$ and $\left\{s_k^{(i)}\right\}_{k\in\mathbb{N}}$ for all $i \in \{1, ..., n_s\}$, as follows: $\bar{\tau}_k = \theta\tau_k$, and $\bar{s}_k^{(i)} = \theta s_k^{(i)}$ for all $k \in \mathbb{N}$. Henceforth, variables $\tilde{z}$, $\tilde{x}$, $\tilde{\varepsilon}$, $\tilde{S}$, u are denoted $\bar{z}$, $\bar{x}$, $\bar{\varepsilon}$, $\bar{S}$, $\bar{u}$ (respectively) when expressed in the $\bar{\tau}$ time frame —i.e. $\bar{z}(\bar{\tau}) = \tilde{z}(\tau(\bar{\tau})) = \tilde{z}(\bar{\tau}/\theta)$. Also, the notation $\bar{z}_k^{(+)}$ bears the same meaning as before, but *w.r.t.* the subdivisions $\{\bar{\tau}_k\}_{k\in\mathbb{N}}$ and $\left\{\bar{s}_k^{(i)}\right\}_{k\in\mathbb{N}}$.

In this new time scale, the observer is given by the set of equations ($\bar{\mathcal{O}}_1$)-($\bar{\mathcal{O}}_2$):

$$\begin{cases} \dfrac{d\bar{z}(\bar{\tau})}{d\bar{\tau}} & = A(\bar{u})\bar{z}(\bar{\tau}) + \frac{1}{\theta}\tilde{b}(\bar{z}, \bar{u}) \\ \dfrac{d\bar{S}(\bar{\tau})}{d\bar{\tau}} & = -\left(A(\bar{u}) + \frac{1}{\theta}D_{\tilde{x}}\tilde{b}(\bar{z}, \bar{u})\right)' \bar{S} - \bar{S}\left(A(\bar{u}) + \frac{1}{\theta}D_{\tilde{x}}\tilde{b}(\bar{z}, \bar{u})\right) - \bar{S}Q\bar{S} \end{cases} \tag{$\bar{\mathcal{O}}_1$}$$

$$\begin{cases} \bar{z}_k^{(+)} = & \bar{z}_k^{(-)} - \bar{S}_k^{(+)^{-1}} \sum_{i \in \sigma_k} C^{(\mathfrak{s}_i)'} R^{(\mathfrak{s}_i)^{-1}} C^{(\mathfrak{s}_i)} \left(\bar{z}_k^{(-)} - \bar{x}(\bar{\tau}_k)\right) \left(\bar{s}_{l_k^{(i)}}^{(i)} - \bar{s}_{l_k^{(i)}-1}^{(i)}\right) \\ \bar{S}_k^{(+)} = & \bar{S}_k^{(-)} + \sum_{i \in \sigma_k} C^{(\mathfrak{s}_i)'} R^{(\mathfrak{s}_i)^{-1}} C^{(\mathfrak{s}_i)} \left(\bar{s}_{l_k^{(i)}}^{(i)} - \bar{s}_{l_k^{(i)}-1}^{(i)}\right) \end{cases} \tag{$\bar{\mathcal{O}}_2$}$$

Exponential convergence

The rest of the proof is based on a Lyapunov function argument, the candidate function being $V(\bar{\varepsilon}) = (\bar{\varepsilon}'\bar{S}\bar{\varepsilon})(\bar{\tau})$. Provided that $\bar{S}(\bar{\tau})$ remains *s.p.d.* then, $V(\bar{\varepsilon}) > 0$ for all $\bar{\varepsilon} \neq 0_{\mathbb{R}^n}$. In the sequel, after stating a theorem that ensures the stability of the matrix $\bar{S}$, we compute the time derivative of $V(\bar{\varepsilon})$ in order to display the exponential convergence of the proposed observer.

Theorem 3.

Let us consider the asynchronous, continuous-discrete, Riccati equation of observer ($\bar{\mathcal{O}}_1$)-($\bar{\mathcal{O}}_2$), that is to say, with $\mathcal{A} = A(\bar{u}) + \frac{1}{\theta} D_{\tilde{x}}\tilde{b}(\bar{z}, \bar{u})$:

$$\begin{cases} \dfrac{d\bar{S}(\bar{\tau})}{d\bar{\tau}} = & -\mathcal{A}'\bar{S} - \bar{S}\mathcal{A} - \bar{S}Q\bar{S} \\ \bar{S}_k^{(+)} = & \bar{S}_k^{(-)} + \displaystyle\sum_{i\in\sigma_k} C^{(\mathfrak{s}_i)'} R^{(\mathfrak{s}_i)^{-1}} C^{(\mathfrak{s}_i)} \left(\bar{s}^{(i)}_{l_k^{(i)}} - \bar{s}^{(i)}_{l_k^{(i)}-1}\right) \end{cases} \tag{4}$$

Where $(\mathcal{A}, C)$ is a time-dependent observable pair.

We assume that all the elements of the matrices $\mathcal{A}$ and C belongs to $L^\infty([0,T], \mathbb{R})$, and are uniformly bounded w.r.t. the L^∞ norm by some positive scalar $B > 0$. Moreover, all the elements of C are differentiable at least once and have their derivatives bounded over time.

Then, $\bar{S}(\bar{\tau})$ is well defined and it is s.p.d. for all times. Moreover, for all $\bar{T} > 0$, there exist constants $\mu_i > 0$, $i \in \{1, ..., n_s\}$, and $0 < \alpha < \beta$, such that, for all subdivisions $\left\{\bar{s}_k^{(i)}\right\}_{k\in\mathbb{N}}$ with $\left(\bar{s}_k^{(i)} - \bar{s}_{k-1}^{(i)}\right) \leq \mu_i$, we have:

$$\alpha Id \leq \bar{S}(\bar{\tau}) \leq \beta Id \quad \textit{for all} \quad \bar{\tau} \geq \bar{T}$$

The constants α and β are independent of θ and the shape of the subdivisions.

Proof. The proof is detailed in Appendix A. □

Let us now resume the convergence study with the computation of $\frac{d}{d\bar{\tau}}V(\bar{\varepsilon})$:

$$\frac{d\bar{\varepsilon}}{d\bar{\tau}}(\bar{\tau}) = \frac{d}{d\bar{\tau}}(\bar{z} - \bar{x})(\bar{\tau}) = A(\bar{u})\bar{\varepsilon} + \frac{1}{\theta}\left(b(\bar{z}, \bar{u}) - b(\bar{x}, \bar{u})\right) \tag{5}$$

$$\begin{aligned} \frac{dV}{d\bar{\tau}}(\bar{\varepsilon}) &= \frac{d(\bar{\varepsilon}'\bar{S}\bar{\varepsilon})}{d\bar{\tau}}(\bar{\tau}) \\ &= \frac{2}{\theta}\left(\bar{\varepsilon}'\bar{S}\right)\left(\tilde{b}(\bar{z}, \bar{u}) - \tilde{b}(\bar{x}, \bar{u}) - D_{\tilde{x}}\tilde{b}(\bar{z}, \bar{u})\bar{\varepsilon}\right) - \left(\bar{\varepsilon}'\bar{S}Q\bar{S}\bar{\varepsilon}\right) \end{aligned} \tag{6}$$

Next, we determine the expression of $V_k^{(+)}$ —i.e. $V(\bar{\tau}_k)$ after a prediction step:

$$\begin{aligned} \bar{\varepsilon}_k^{(+)} &= \bar{z}_k^{(+)} - \bar{x}(\bar{\tau}_k) \\ &= \left[Id - \bar{S}_k^{(+)^{-1}} \sum_{i\in\sigma_k} C^{(\mathfrak{s}_i)'} R^{(\mathfrak{s}_i)^{-1}} C^{(\mathfrak{s}_i)} \left(\bar{s}^{(i)}_{l_k^{(i)}} - \bar{s}^{(i)}_{l_k^{(i)}-1}\right)\right] \bar{\varepsilon}_k^{(-)} \end{aligned}$$

On the other hand, in $\bar{\tau}_k$, we have:

$$\begin{aligned} V_k^{(+)} &= (\bar{\varepsilon}'\bar{S}\bar{\varepsilon})_k^{(+)} \\ &= \bar{\varepsilon}_k^{(-)'}\left[\bar{S}_k^{(+)} - 2\mathcal{M} + \mathcal{M}\bar{S}_k^{(+)^{-1}}\mathcal{M}\right]\bar{\varepsilon}_k^{(-)} \end{aligned} \tag{7}$$

where

$$\mathcal{M} = \sum_{i\in\sigma_k} C^{(\mathfrak{s}_i)'}R^{(\mathfrak{s}_i)^{-1}}C^{(\mathfrak{s}_i)}\left(\bar{s}^{(i)}_{l_k^{(i)}} - \bar{s}^{(i)}_{l_k^{(i)}-1}\right) = \bar{S}_k^{(+)} - \bar{S}_k^{(-)} \tag{8}$$

In (7), the matrix $\mathcal{M}$ is replaced by the right-hand side of (8). Simplifications lead to:

$$V_k^{(+)} = \bar{\varepsilon}_k^{(-)'}\left[\bar{S}_k^{(-)^{-1}}\bar{S}_k^{(+)}\bar{S}_k^{(-)^{-1}}\right]^{-1}\bar{\varepsilon}_k^{(-)}$$

Using ($\mathcal{O}_2$) again allows us to write:

$$\begin{aligned} \bar{S}_k^{(-)^{-1}}\bar{S}_k^{(+)}\bar{S}_k^{(-)^{-1}} &= \bar{S}_k^{(-)^{-1}}\left[\bar{S}_k^{(-)} + \sum_{i\in\sigma_k} C^{(\mathfrak{s}_i)'}R^{(\mathfrak{s}_i)^{-1}}C^{(\mathfrak{s}_i)}\left(\bar{s}^{(i)}_{l_k^{(i)}} - \bar{s}^{(i)}_{l_k^{(i)}-1}\right)\right]\bar{S}_k^{(-)^{-1}} \\ &= \bar{S}_k^{(-)^{-1}} + \bar{S}_k^{(-)^{-1}}\underbrace{\sum_{i\in\sigma_k} C^{(\mathfrak{s}_i)'}R^{(\mathfrak{s}_i)^{-1}}C^{(\mathfrak{s}_i)}\left(\bar{s}^{(i)}_{l_k^{(i)}} - \bar{s}^{(i)}_{l_k^{(i)}-1}\right)}_{(\star)}\bar{S}_k^{(-)^{-1}} \end{aligned}$$

Before going any further, let us remind the matrix inversion lemma.

Lemma 4 (Matrix inversion lemma)**.**
Let M be a s.p.d. matrix and R an invertible matrix. then

$$(M + MC'R^{-1}CM)^{-1} = M^{-1} - C'(R + CMC')^{-1}C$$

In order to use this lemma, it is necessary to express the sum of matrices that appears in expression ($\star$) as a product of matrices. First, let us denote:

$$R_{\sigma_k} = diag\left[\left\{R^{(\mathfrak{s}_i)} : i\in\sigma_k\right\}\right] \text{ and } I_{\sigma_k} = diag\left[\left\{\left(\bar{s}^{(i)}_{l_k^{(i)}} - \bar{s}^{(i)}_{l_k^{(i)}-1}\right)Id : i\in\sigma_k\right\}\right]$$

Then the sum in ($\star$) can be expressed using the block matrices:

$$\sum_{i\in\sigma_k} C^{(\mathfrak{s}_i)'}R^{(\mathfrak{s}_i)^{-1}}C^{(\mathfrak{s}_i)}\left(\bar{s}^{(i)}_{l_k^{(i)}} - \bar{s}^{(i)}_{l_k^{(i)}-1}\right) = C'_{\sigma_k}R^{-1}_{\sigma_k}I_{\sigma_k}C_{\sigma_k}$$

By definition, I_{σ_k} is invertible and using[3] Lemma 4:

$$\begin{aligned} \left[\bar{S}_k^{(-)^{-1}}\bar{S}_k^{(+)}\bar{S}_k^{(-)^{-1}}\right]^{-1} &= \left[\bar{S}_k^{(-)^{-1}} + \bar{S}_k^{(-)^{-1}}C'_{\sigma_k}R^{-1}_{\sigma_k}I_{\sigma_k}C_{\sigma_k}\bar{S}_k^{(-)^{-1}}\right]^{-1} \\ &= \bar{S}_k^{(-)} - C'_{\sigma_k}\left(R_{\sigma_k}I^{-1}_{\sigma_k} + C_{\sigma_k}\bar{S}_k^{(-)^{-1}}C'_{\sigma_k}\right)^{-1}C_{\sigma_k} \end{aligned}$$

[3]Note that, matrices R_{σ_k} and $I^{-1}_{\sigma_k}$ do commute. Indeed, by definition, each blocks of R_{σ_k} correspond to a block of $I^{-1}_{\sigma_k}$ made of an identity matrix times some constant parameter.

Therefore, we obtain the following system, for all $k \in \mathbb{N}$:

$$\begin{cases} \dfrac{dV}{d\bar{\tau}}(\bar{\varepsilon}) = & \frac{2}{\theta}(\bar{\varepsilon}'\bar{S})[\tilde{b}(\bar{z},\bar{u}) - \tilde{b}(\bar{x},\bar{u}) - D_{\bar{x}}\tilde{b}(\bar{z},\bar{u})\bar{\varepsilon}] - (\bar{\varepsilon}'\bar{S}Q\bar{S}\bar{\varepsilon}) \qquad \text{for } \bar{\tau} \in [\bar{\tau}_{k-1}, \bar{\tau}_k[\\ V_k^{(+)} = & (\bar{\varepsilon}'\bar{S}\bar{\varepsilon})_k^{(-)} - \bar{\varepsilon}_k^{(-)'} C'_{\sigma_k} \left(R_{\sigma_k} I_{\sigma_k}^{-1} + C_{\sigma_k} \bar{S}_k^{(-)^{-1}} C'_{\sigma_k} \right)^{-1} C_{\sigma_k} \bar{\varepsilon}_k^{(-)} \qquad \text{for } \bar{\tau} = \bar{\tau}_k \end{cases}$$

Since $\tilde{b}(.,\bar{u})$ is Lipschitz in its first argument (uniformly *w.r.t* $\bar{u}$) and $D_x\tilde{b}(.,\bar{u})$ is upper bounded:

$$\begin{aligned} \|\tilde{b}(\bar{z},\bar{u}) - \tilde{b}(\bar{x},\bar{u}) - D_{\bar{x}}\tilde{b}(\bar{z},\bar{u})\bar{\varepsilon}\| &\leq L_b\|\bar{z} - \bar{x}\| + L_b\|\bar{\varepsilon}\| \\ &= 2L_b\|\bar{\varepsilon}\| \end{aligned}$$

Theorem 3 provides bounds for $\bar{S}$ for times greater than a fixed $\bar{T} > 0$, and the constraints on Q are given in (1), thus leading to:

$$\frac{dV}{d\bar{\tau}}(\bar{\varepsilon}) \leq \left(\frac{4}{\theta}\frac{\beta}{\alpha}L_b - \alpha\underline{q} \right) (\bar{\varepsilon}'\bar{S}\bar{\varepsilon})$$

Since $\bar{S}(\bar{\tau})$ is positive definite, the derivative of $V(\bar{\varepsilon})$ is negative for θ chosen such that $\frac{4}{\theta}\frac{\beta}{\alpha}L_b - \alpha\underline{q} < 0$. Furthemore, we easily show that $V_k^{(+)} \leq V_k^{(-)}$ for all $k \in \mathbb{N}$. Indeed:

$$V_k^{(+)} = V_k^{(-)} - \bar{\varepsilon}_k^{(-)'} \underbrace{C_{\sigma_k}{}' \left(R_{\sigma_k} I_{\sigma_k}^{-1} + C_{\sigma_k} \bar{S}_k^{(-)^{-1}} C_{\sigma_k}{}' \right)^{-1} C_{\sigma_k}}_{(\diamond)} \bar{\varepsilon}_k^{(-)}$$

Since $\bar{S}(\bar{\tau})$ and $R_{\sigma_k} I_{\sigma_k}^{-1}$ are *s.p.d.* for all times, then the matrix $(\diamond)$ is at least positive semidefinite and:

$$V_k^{(+)} = (\bar{\varepsilon}'\bar{S}\bar{\varepsilon})_k^{(+)} \leq (\bar{\varepsilon}'\bar{S}\bar{\varepsilon})_k^{(-)} \quad \text{for all} \quad k \in \mathbb{N} \tag{9}$$

This shows the asymptotic convergence of the observer. Moreover, this convergence is exponential. Indeed, let $\bar{\tau}_k^\star = \min_{k\in\mathbb{N}} \left\{ \bar{\tau}_k : \bar{\tau}_k > \bar{T} \right\}$, then for $\bar{\tau} \in]\bar{T}, \bar{\tau}_k^\star[$:

$$\begin{aligned} (\bar{\varepsilon}'\bar{S}\bar{\varepsilon})(\bar{\tau}) &= (\bar{\varepsilon}'\bar{S}\bar{\varepsilon})(\bar{T}) + \left(\frac{4}{\theta}\frac{\beta}{\alpha}L_b - \alpha\underline{q} \right) \int_{\bar{T}}^{\bar{\tau}} (\bar{\varepsilon}'\bar{S}\bar{\varepsilon})(v)dv \\ &\leq (\bar{\varepsilon}'\bar{S}\bar{\varepsilon})(\bar{T}) e^{(L_b \frac{4}{\theta}\frac{\beta}{\alpha} - \alpha\underline{q})(\bar{\tau} - \bar{T})} \end{aligned} \tag{10}$$

where (10) has been obtained by using Grönwall's lemma.
For $\bar{\tau} \in [\bar{\tau}_k, \bar{\tau}_{k+1}[$, inequality (10) is true with $\bar{\tau}_k$ replacing $\bar{T}$. Then, using relation (9) we show by iteration that (10) is in fact true for all $\bar{\tau} > \bar{T}$, independently from the subdivisions $\left\{ \bar{s}_k^{(i)} \right\}_{k\in\mathbb{N}}$ and $\left\{ \bar{\tau}_k^{(i)} \right\}_{k\in\mathbb{N}}$.

Since $\bar{\varepsilon}(\bar{\tau}) = \tilde{\varepsilon}(\tau)$, then inequality (10) becomes:

$$\|\tilde{\varepsilon}(\tau)\|^2 \leq \frac{\beta}{\alpha} \left\| \tilde{\varepsilon}\left(\frac{\bar{T}}{\theta} \right) \right\|^2 e^{\left(4L_b \frac{\beta}{\alpha} - \theta\alpha\underline{q}\right)(\tau - \frac{\bar{T}}{\theta})} \quad \text{for all} \quad \tau > \frac{\bar{T}}{\theta}$$

Finally, following the definition of $\varepsilon(\tau) = \Delta^{-1}\tilde{\varepsilon}(\tau)$, since $\|\Delta^{-1}\| \leq \theta^{n^*-1}$ and $\|\Delta\| \leq 1$, we conclude that

$$\|\varepsilon(\tau)\|^2 \leq \theta^{2(n^*-1)} \frac{\beta}{\alpha} \left\| \varepsilon\left(\frac{\bar{T}}{\theta}\right) \right\|^2 e^{\left(4L_b \frac{\beta}{\alpha} - \theta\alpha\underline{q}\right)(\tau - \frac{\bar{T}}{\theta})} \quad \text{for all} \quad \tau > \frac{\bar{T}}{\theta} \tag{11}$$

5 Illustrative example

Let us consider a boat[4] moving in an area delimited by two beacons (denoted by A and B). The state of this system is $x = (x_1, x_2, \gamma)' \in \mathbb{R}^3$, where as it is schematised in Figure 1a, $(x_1, x_2)' \in \mathbb{R}^2$ is the position of the boat *w.r.t.* the reference frame attached to A, and $\gamma \in \mathbb{R}$ is the orientation of the boat, that is the angle formed by axes $\mathcal{X}_1^r$ and $\mathcal{X}_1^b$ —this latter axis defining a reference frame attached to the boat.

Both beacons emit a signal that is detected by an onboard rotational position sensor. This sensor consists of a rotating oriented cavity that guides the signal to the actual electronic sensor. Therefore, the signal emitted by a given beacon is detected when the oriented cavity is aligned with the boat-beacon line[5]. This mechanism provides a measurement of the angle formed by axis $\mathcal{X}_1^b$ and the boat-beacon line. Furthermore, we assume that the signal received from A also provides the distance between A and the boat.

The dynamics are simply modelled by the following control system:

$$\begin{cases} \dot{x}_1(\tau) = & v(\tau)\cos(\gamma(\tau)) \\ \dot{x}_2(\tau) = & v(\tau)\sin(\gamma(\tau)) \\ \dot{\gamma}(\tau) = & u(\tau) \end{cases} \tag{Σ_{Boat}}$$

where $u(\tau)$ and $v(\tau)$ are the controls.

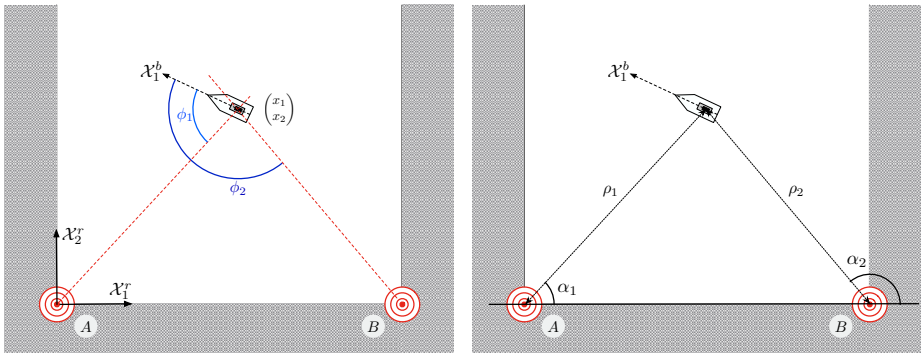
In order to deal with simple equations for the output vector, the system is rewritten using polar coordinates *w.r.t.* both A and B —cf. Figure 1b. Let x_B be the abscissa of B in the $(\mathcal{X}_1^r, \mathcal{X}_2^r)$ reference frame. Then, the angles α_1, α_2 and distances ρ_1, ρ_2 are such that:

- $x_1 = \rho_1 \cos(\alpha_1)$ and $x_2 = \rho_1 \sin(\alpha_1)$;
- $x_1 = x_B + \rho_2 \cos(\alpha_2)$ and $x_2 = \rho_2 \sin(\alpha_2)$.

In those new coordinates, the full dynamics, with output vector $y(t)$, are

[4] Or a wheeled mobile robot.

[5] Boat-beacon line: the line that passes through $(x_1, x_2)'$ and the center of the concerned beacon.



(a) Notations (planar representation). (b) Notations (polar representations).

Figure 1

given by

$$\begin{cases} \dot{\alpha}_1 = & \frac{v}{\rho_1}\sin(\gamma-\alpha_1) \\ \dot{\rho}_1 = & v\cos(\gamma-\alpha_1) \\ \dot{\alpha}_2 = & \frac{v}{\rho_2}\sin(\gamma-\alpha_2) \\ \dot{\rho}_2 = & v\cos(\gamma-\alpha_2) \\ \dot{\gamma} = & u \end{cases} \quad \text{and} \quad y(t) = \begin{pmatrix} \phi_1 \\ \rho_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \pi+\alpha_1-\gamma \\ \rho_1 \\ \pi+\alpha_2-\gamma \end{pmatrix} \tag{12}$$

With the help of system (12), the original system (Σ_{Boat}) is observable since we can find a change of coordinates that puts (Σ_{Boat}) under a normal observability form —i.e. system (13). Actually, the exact position of the boat can be computed from the knowledge of $\phi_2-\phi_1$ and ρ_1. The orientation of the boat can then be easily deduced. However, without both angle measurements observability is lost. Since the rotational position sensor makes the angle measurements asynchronous, it makes this example appropriate.

The outputs of System (12) are decomposed into two sensors having non-uniform sampling times: $\mathfrak{s}_1 = \{1,2\}$ and $\mathfrak{s}_2 = \{3\}$. Indeed, times when the rotational cavity is aligned with a given beacon depend on the state of the boat.

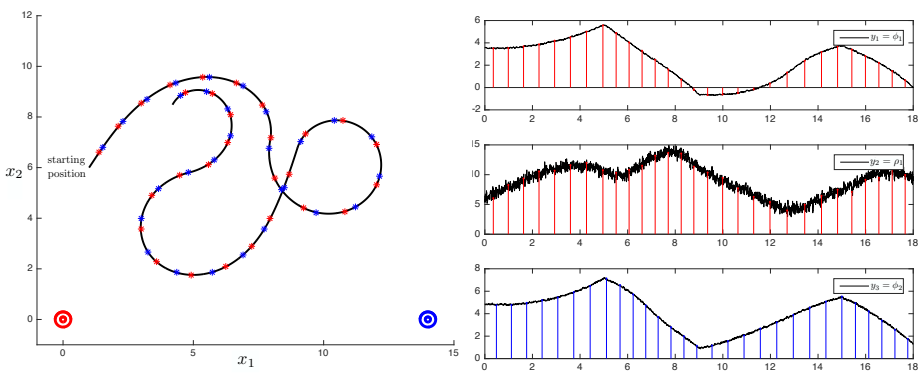
In order to apply the asynchonous high-gain Kalman filter under discussion, we put (12) under the normal observability form below by defining $z(\tau) = y(\tau)$.

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = \begin{pmatrix} \dfrac{v\sin(z_1)}{z_2} - u \\ -v\cos(z_1) \\ \dfrac{v\sin(z_3)}{\bar{\rho}(z_1,z_2,z_3)} - u \end{pmatrix} \tag{13}$$

where $\bar{\rho}(z_1,z_2,z_3) = z_2\cos(z_3-z_1) + \sqrt{x_B^2 - z_2^2}\sin^2(z_3-z_1)$ is the expression of ρ_2 (as a function of $\phi_2-\phi_1 = z_3-z_1$ and $\rho_1 = z_2$) obtained through the law of cosines.

We considered the boat trajectory shown in Figure 2a. Here, the boat's speed (i.e. $v(\tau)$) is kept constant except for $\tau \in [5, 10]$ where it is momentarily raised to a higher constant value. The initial state of system is $x(0) = (1, 6, 1)$ and the initial state of the observer is directly set in the normal coordinates[6]. The Riccati equation's initial datum is set by solving an algebraic Riccati equation using the informations available at time $t = 0$.

The estimated trajectory, compared to the actual boat trajectory is shown in Figure 3. Figure 3a highlights the increased convergence speed due to a large high-gain parameter. Let us remark that when the high-gain parameter equals 1, the displayed observer fails to achieve convergence. Figure 3b shows the performance of the observer when additive noise is introduced in the output. We used a gaussian noise[7] colored through a first order discrete filter, as illustrated in Figure 2b. Because of the known sensitivity of high-gain observers with respect to noise, only the lower value of the high-gain parameter was considered in the second experiment. This tradeoff between convergence efficiency and robustness *w.r.t.* measurement noise could be further investigated with the use of an adaptive scheme for the high-gain parameter in the spirit of [9, 16, 38].



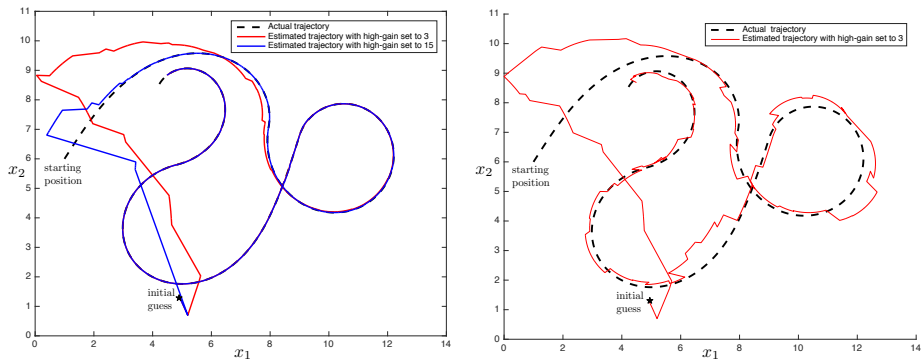
(a) Boat trajectory showing the position at sampling times.

(b) System's output signal with additive noise. The markers highlight sampling times.

Figure 2: Informations relative to the first (resp. second) beacon appear in red (resp. blue)

[6] As a consequence, the initial guess lacks consistency *w.r.t.* the problem's physics which makes the task harder for the observer.

[7] Having its standard deviation equals to 0.1 for the angle measurements, and 1 for the distance measurements.



(a) Trajectory estimation without measurement noise.

(b) Trajectory estimation with additive noise on the measurement —cf. Figure 2b.

Figure 3

6 Conclusion

In this paper, a high-gain extended Kalman filter for nonlinear continuous-discrete systems with multirate sampled outputs has been presented and its global asymptotical convergence, proved. The proposed design consists of two steps: (i) an open loop prediction when no measurements are available, and (ii) an impulsive correction as soon as new measurements are available. To this end, each correction step involves a weighted sum of the output errors calculated on the basis of the measurements available at this sample time. In order to better handle possible cross-correlations between measurements always available at the same time, *sensors* are defined as subsets of the output vector. Moreover, the Riccati matrix of the observer is shown to be bounded from above and below provided that (Σ_c), the underlying continuous system, is observable and for small enough sampling intervals.

Some improvements are left for the future. First of all, as it is illustrated in the example, the well known sensitivity of the high-gain design to measurement noise could be addressed with the help of an adaptive scheme in the spirit of [9, 16]. An approach taking into account several high-gain parameters instead of one only (i.e. one parameter per virtual sensor) in the spirit of [38] is another possible extension to the present work. The present study can also be conducted in the framework of hybrid systems, cf. [20], as is it done for synchronous hybrid systems in [32].

Finally, the presence of redundant sensors can lead to an improved version of the proposed design. Indeed, the maximum step size condition on the time subdivision of a given sensor could be relaxed provided there is an active redundant sensor —for example in submarine robotics the vehicle's speed available from a surface GPS is lost when the robot dives but can be obtained again (computed with respect to the ground) via a Doppler velocity log.

A Bounds for the solution of the Riccati equation

This section is dedicated to the proof of Theorem 3. It follows the structure of [7] where a similar result is proved for synchronous continuous-discrete systems. Although the present proof shares the same structure, differences specific to the asynchronous setting make this exposure necessary. However, only proofs having notable differences are detailed.

The complete argument of Theorem 3 is divided into two parts.

- In a first part, for a given $T^\star > 0$, we prove the existence of an upper bound for times greater than $T^\star$. Here, the argument mainly relies on the regularity of $\bar{S}$, and the bound depends on the maximum step size of the subdivision, $\{\bar{\tau}_k\}_{k\in\mathbb{N}}$ regardless of the underlying subdivisions $\left\{\bar{s}_k^{(i)}\right\}_{k\in\mathbb{N}}$.
- In a second time, we prove the existence of a lower bound for times greater than $T_\star > T^\star$. In this second part, the result relies on the observability of the underlying continuous system (Σ_c), and requires small enough maximum time steps for each subdivision $\left\{\bar{s}_k^{(i)}\right\}_{k\in\mathbb{N}}$.

The quantity $\bar{T}$ that appears in Theorem 3 is simply $\bar{T} = T_\star$.

In the following, we assume the existence of a positive definite solution $\bar{S}(\bar{\tau})$ on a small interval of time, which is ensured by the Sylvester criterion. We later on show that this interval of time is actually $\mathbb{R}^+$.

A.1 Upper bound

In order to prove that $\bar{S}_k^{(+)}$ is upper bounded for times greater than $T^\star$, we should remember that if $\bar{S}$ is a symmetric positive semidefinite matrix, then we have $\bar{S} \leq Tr(\bar{S})Id$.

Lemma 5. [7].
Let $\bar{S} : [0, T[\to S_n$ be a solution to $\frac{d\bar{S}}{d\bar{\tau}} = -\mathcal{A}'\bar{S} - \bar{S}\mathcal{A} - \bar{S}Q\bar{S}$, then for almost all $\bar{\tau} \in [0, T[$:

$$\frac{d}{d\bar{\tau}}Tr(\bar{S}) \leq -a(Tr(\bar{S}(\bar{\tau})))^2 + 2bTr(\bar{S}(\bar{\tau})) \quad \textit{where} \begin{cases} a = \frac{\lambda_{min}(Q)}{n} \\ b = \sup_{\bar{\tau}} Tr\left(\mathcal{A}'(\bar{\tau})\mathcal{A}(\bar{\tau})\right)^{\frac{1}{2}} \end{cases}$$

Lemma 6. [7].
Let a, b be two positive constants. Let $x : [0, T[\to \mathbb{R}^+$ (possibly $T = +\infty$) be an absolutely continuous function satisfying for almost all $0 < \tau < T$ the inequality:

$$\dot{x}(\tau) \leq -ax^2(\tau) + 2bx(\tau)$$

The roots of $-aX^2 + 2bX$ are $\frac{2b}{a}$ and 0. The solution $x(\tau)$ is such that:

$$x(\tau) \leq \max\left\{x(0), \tfrac{2b}{a}\right\} \textit{ for all } \tau \in [0, T[$$

In addition if $x(0) > \frac{2b}{a}$ *then for all* $\tau > 0 \in [0, T[$ *we have the two inequalities:*

$$x(\tau) \leq \frac{2b}{a} + \frac{2b}{a}\frac{1}{e^{2b\tau} - 1} \tag{14}$$

$$x(\tau) \leq \frac{2bx_0 e^{2b\tau}}{ax_0\left(e^{2b\tau} - 1\right) + 2b} \tag{15}$$

Let us denote $r = \sup_{i,k}\left(Tr\left(C^{(\mathfrak{s}_i)'}R^{(\mathfrak{s}_i)^{-1}}C^{(\mathfrak{s}_i)}\right)\right)$. According to equation (4) and to the previous lemmas, upper bounding $\bar{S}$ turns into proving that $x_k^{(+)}$, solution of

$$\begin{cases} \frac{dx}{d\tau} & = & -ax^2 + 2bx \\ x_k^{(+)} & = & x_k^{(-)} + r\sum_{i\in\sigma_k}\left(\bar{s}^{(i)}_{l_1^{(i)}} - \bar{s}^{(i)}_{l_k^{(i)}-1}\right) \end{cases} \tag{16}$$

is bounded for all $\bar{\tau}_k > T^\star$, $k \in \mathbb{N}$, independently from the chosen subdivisions. It leads us to Lemma 7.

Lemma 7.

The solution of (16) *is such that:*

$$x(\bar{\tau}) \leq \frac{2b}{a} + \frac{2b}{a}\frac{1}{e^{2b\bar{\tau}} - 1} + n_s\, r\bar{\tau}$$

for any $\bar{\tau} > 0$*, before or after a discrete step.*

Proof. Bound (14) gives:

$$x_1^{(+)} \leq \frac{2b}{a} + \frac{2b}{a}\frac{1}{e^{2b\bar{\tau}_1} - 1} + r\sum_{i\in\sigma_1}\left(\bar{s}^{(i)}_{l_1^{(i)}} - \bar{s}^{(i)}_{l_1^{(i)}-1}\right)$$

We denote $\chi_k = r\sum_{j=1}^{k}\sum_{i\in\sigma_k}\left(\bar{s}^{(i)}_{l_k^{(i)}} - \bar{s}^{(i)}_{l_k^{(i)}-1}\right)$ and the previous inequality trivially becomes

$$x_1^{(+)} \leq \frac{2b}{a} + \frac{2b}{a}\frac{1}{e^{2b\bar{\tau}_1} - 1} + \chi_1 \tag{17}$$

We remark that (17) leads to the inequality below:

$$x_1^{(+)} \leq \frac{2b}{a} + \frac{2b}{a}\frac{1}{e^{2b\bar{\tau}_1} - 1} + n_s\, r\,\bar{\tau}_1 \tag{18}$$

Let us now generalise this last inequality for all $k \in \mathbb{N}$. However, in order to do so, it is necessary to manipulate inequalities shaped as (17) instead of (18). Let us now rewrite bound (15) as follows:

$$x_2^{(-)} \leq \frac{2b}{a} + \frac{2b}{a}\frac{x_1^{(+)} - \frac{2b}{a}}{x_1^{(+)}\left(e^{2b(\bar{\tau}_2 - \bar{\tau}_1)} - 1\right) + \frac{2b}{a}} = \frac{2bx_1^{(+)}e^{2b(\bar{\tau}_2 - \bar{\tau}_1)}}{ax\left(e^{2b(\bar{\tau}_2 - \bar{\tau}_1)} - 1\right) + 2b} \tag{19}$$

We want to replace $x_1^{(+)}$ by the upper bound found in (17).
Let us define the function

$$h(x) = \frac{2bxe^{2b\bar{\tau}}}{ax\left(e^{2b\bar{\tau}}-1\right)+2b}$$

whose derivative *w.r.t.* x is

$$\begin{aligned} h^{'}(x) &= \frac{e^{2b\bar{\tau}}2b\left[ax(e^{2b\bar{\tau}}-1)+2b\right]-a(e^{2b\bar{\tau}}-1)xe^{2b\bar{\tau}}2b}{\left[ax(e^{2b\bar{\tau}}-1)+2b\right]^2} \\ &= \frac{e^{2b\bar{\tau}}(2b)^2}{\left[ax(e^{2b\bar{\tau}}-1)+2b\right]^2} \end{aligned}$$

Since the derivative is positive for all $\bar{\tau} > 0$, we can replace $x_1^{(+)}$ by its upper bound in (19):

$$\begin{aligned} x_2^{(-)} &\leq \frac{2b}{a} + \frac{2b}{a}\frac{[\frac{2b}{a}+\frac{2b}{a}\frac{1}{e^{2b\bar{\tau}_1}-1}+\chi_1]-\frac{2b}{a}}{[\frac{2b}{a}+\frac{2b}{a}\frac{1}{e^{2b\bar{\tau}_1}-1}+\chi_1](e^{2b(\bar{\tau}_2-\bar{\tau}_1)}-1)+\frac{2b}{a}} \\ &\leq \frac{2b}{a} + \frac{2b}{a}\frac{2b}{a}\frac{1}{e^{2b\bar{\tau}_1}-1}\frac{1}{[\frac{2b}{a}+\frac{2b}{a}\frac{1}{e^{2b\bar{\tau}_1}-1}+\chi_1](e^{2b(\bar{\tau}_2-\bar{\tau}_1)}-1)+\frac{2b}{a}} \\ &\qquad + \frac{2b}{a}\frac{\chi_1}{\left[\frac{2b}{a}+\frac{2b}{a}\frac{1}{e^{2b\bar{\tau}_1}-1}+\chi_1\right](e^{2b(\bar{\tau}_2-\bar{\tau}_1)}-1)+\frac{2b}{a}} \end{aligned}$$

We lower bound the denominator of the last term with:

$$[\frac{2b}{a}+\frac{2b}{a}\frac{1}{e^{2b\bar{\tau}_1}-1}+\chi_1](e^{2b(\bar{\tau}_2-\bar{\tau}_1)}-1)+\frac{2b}{a} \geq \frac{2b}{a}$$

and the denominator of the second term with:

$$[\frac{2b}{a}+\frac{2b}{a}\frac{1}{e^{2b\bar{\tau}_1}-1}+\chi_1](e^{2b(\bar{\tau}_2-\bar{\tau}_1)}-1)+\frac{2b}{a} \geq [\frac{2b}{a}+\frac{2b}{a}\frac{1}{e^{2b\bar{\tau}_1}-1}](e^{2b(\bar{\tau}_2-\bar{\tau}_1)}-1)+\frac{2b}{a}$$

We also simplify $(2b/a)$ in those two terms:

$$\begin{aligned} x_2^{(-)} &\leq \frac{2b}{a} + \frac{2b}{a}\frac{1}{e^{2b\bar{\tau}_1}-1}\frac{1}{([1+\frac{1}{e^{2b\bar{\tau}_1}-1}](e^{2b(\bar{\tau}_2-\bar{\tau}_1)}-1)+1)} + \chi_1 \\ &\leq \frac{2b}{a} + \frac{2b}{a}\frac{1}{e^{2b\bar{\tau}_2}-e^{2b\bar{\tau}_1}+e^{2b\bar{\tau}_1}-1} + \chi_1 \end{aligned}$$

Thus we have:

$$\begin{aligned} x_2^{(+)} &\leq \frac{2b}{a} + \frac{2b}{a}\frac{1}{e^{2b\bar{\tau}_2}-1} + \chi_1 + r\sum_{i\in\sigma_2}\left(\bar{s}^{(i)}_{l_2^{(i)}} - \bar{s}^{(i)}_{l_2^{(i)}-1}\right) \\ &\leq \frac{2b}{a} + \frac{2b}{a}\frac{1}{e^{2b\bar{\tau}_2}-1} + \chi_2 \end{aligned}$$

$$\text{with} \quad \chi_2 = r\left(\sum_{i\in\sigma_1}\left(\bar{s}^{(i)}_{l_1^{(i)}} - \bar{s}^{(i)}_{l_1^{(i)}-1}\right) + \sum_{i\in\sigma_2}\left(\bar{s}^{(i)}_{l_2^{(i)}} - \bar{s}^{(i)}_{l_2^{(i)}-1}\right)\right)$$

Let us notice that for $i \in \sigma_1, \bar{s}^{(i)}_{l^{(i)}_1} = \bar{\tau}_1$, for $i \in \sigma_2, \bar{s}^{(i)}_{l^{(i)}_2} = \bar{\tau}_2$ —and so on and so forth for all $i \in \sigma_k, k \in \mathbb{N}, \bar{s}^{(i)}_{l^{(i)}_k} = \bar{\tau}_k$. At time $\bar{\tau}_2$, for a given sensor $\mathfrak{s}_i$, i belongs to one of the four following subsets:

1. $\sigma_1 \cap \sigma_2$
2. $\sigma_1 \setminus \sigma_2$
3. $\sigma_2 \setminus \sigma_1$
4. $\{1, ..., n_s\} \setminus \sigma_1 \setminus \sigma_2$

We consider first the case $i \in \sigma_1 \cap \sigma_2$, then:

- let $\lambda^{(i)}_2$ be defined as $\lambda^{(i)}_2 = \max\{l \in \mathbb{N} \text{ such that } s^{(i)}_l \leq \tau_2\}$, thus $\lambda^{(i)}_2 = 2$;
- recall that indices $l^{(i)}_k$ are such that

$$\sigma_k = \left\{ i \in \{1, ..., n_s\} | \exists\ l^{(i)}_k \in \mathbb{N} \text{ such that } s^{(i)}_{l^{(i)}_k} = \tau_k \right\}$$

  then, $\bar{s}^{(i)}_{l^{(i)}_2} = \bar{s}^{(i)}_2 = \bar{\tau}_2$, $\bar{s}^{(i)}_{l^{(i)}_2 - 1} = \bar{s}^{(i)}_{l^{(i)}_1} = \bar{s}^{(i)}_1 = \bar{\tau}_1$ and $\bar{s}^{(i)}_{l^{(i)}_1 - 1} = \bar{s}^{(i)}_{l^{(i)}_0} = \bar{\tau}_0 = 0$

- The contribution of sensor $\mathfrak{s}_i$ to χ_2 is of the form:

$$r\left[\left(\bar{s}^{(i)}_{l^{(i)}_1} - \bar{s}^{(i)}_{l^{(i)}_1 - 1}\right) + \left(\bar{s}^{(i)}_{l^{(i)}_2} - \bar{s}^{(i)}_{l^{(i)}_2 - 1}\right)\right] \quad = \quad r\sum_{j=1}^{\lambda^{(i)}_2}\left(\bar{s}^{(i)}_j - \bar{s}^{(i)}_{j-1}\right)$$

In the same way, and dealing with an $i \in \sigma_2 \setminus \sigma_1$, we find:

- $\lambda^{(i)}_2 = 1$, $\bar{s}^{(i)}_{l^{(i)}_2} = \bar{s}^{(i)}_1 = \bar{\tau}_2$, and $\bar{s}^{(i)}_{l^{(i)}_2 - 1} = \bar{s}^{(i)}_0 = 0$;
- In this case, the contribution of sensor $\mathfrak{s}_i$ to χ_2 is of the form:

$$r\left(\bar{s}^{(i)}_{l^{(i)}_2} - \bar{s}^{(i)}_{l^{(i)}_2 - 1}\right) \quad = \quad r\sum_{j=1}^{\lambda^{(i)}_2}\left(\bar{s}^{(i)}_j - \bar{s}^{(i)}_{j-1}\right)$$

By proceeding this way for all the other cases, we find that the contribution of sensor $\mathfrak{s}_i$ to χ_2 is always of the form:

$$\sum_{j=1}^{\lambda^{(i)}_2}\left(\bar{s}^{(i)}_j - \bar{s}^{(i)}_{j-1}\right) \tag{20}$$

and χ_2 can be written as $\chi_2 = r\sum_{i=1}^{n_s}\sum_{j=1}^{\lambda_2^{(i)}}\left(\bar{s}_j^{(i)} - \bar{s}_{j-1}^{(i)}\right) \leq rn_s\bar{\tau}_2$. Therefore,

$$\begin{aligned} x_2^{(+)} &\leq \frac{2b}{a} + \frac{2b}{a}\frac{1}{e^{2b\bar{\tau}_2}-1} + r\sum_{i=1}^{n_s}\sum_{j=1}^{\lambda_2^{(i)}}\left(\bar{s}_j^{(i)} - \bar{s}_{j-1}^{(i)}\right) \\ &\leq \frac{2b}{a} + \frac{2b}{a}\frac{1}{e^{2b\bar{\tau}_2}-1} + rn_s\bar{\tau}_2 \end{aligned} \tag{21}$$

We can generalise by induction (20) and (21) to any $k \in \mathbb{N}\setminus\{0\}$. To do so, we define $\lambda_k^{(i)} = \max\{l \in \mathbb{N} \text{ such that } s_l^{(i)} \leq \tau_k\}$ and, when $\lambda_k^{(i)} = 0$, we use the convention $\sum_{j=1}^{\lambda_k^{(i)}}(\bar{s}_j^{(i)} - \bar{s}_{j-1}^{(i)}) = 0$. This yields

$$\chi_k = r\sum_{j=1}^{k}\sum_{i\in\sigma_k}\left(\bar{s}_{l_k^{(i)}}^{(i)} - \bar{s}_{l_k^{(i)}-1}^{(i)}\right) = r\sum_{i=1}^{n_s}\sum_{j=1}^{\lambda_k^{(i)}}(\bar{s}_j^{(i)} - \bar{s}_{j-1}^{(i)}) \leq rn_s\bar{\tau}_k \tag{22}$$

and $x_k^{(+)} \leq \frac{2b}{a} + \frac{2b}{a}\frac{1}{e^{2b\bar{\tau}_i}-1} + r\,n_s\,\bar{\tau}_k$.

Moreover, we can generalise this inequality to any $\bar{\tau} > 0$, before and after an update.

$$x(\bar{\tau}) \leq \frac{2b}{a} + \frac{2b}{a}\frac{1}{e^{2b\bar{\tau}}-1} + r\,n_s\,\bar{\tau}$$

□

Lemma 8. [7]

Let us define the functions

$$\begin{aligned} \phi(\bar{\tau}) &= \frac{2b}{a} + \frac{2b}{a}\frac{1}{e^{2b\bar{\tau}}-1} + n_s\,r\bar{\tau} \\ \psi_{x_0}(\bar{\tau}) &= \frac{2bx_0e^{2b\bar{\tau}}}{ax_0\left(e^{2b\bar{\tau}}-1\right)+2b} + n_s\,r\bar{\tau} \end{aligned}$$

There exists $\mu_\phi > 0$, and $\mu_\psi(x_0) > 0$ such that $\phi(\bar{\tau})$, respectively $\psi_{x_0}(\bar{\tau})$, is a decreasing function for $\bar{\tau} \in]0, \mu_\phi]$, respectively for $\bar{\tau} \in [0, \mu_\psi(x_0)]$.
Moreover the bound $\mu_\psi(x_0)$ increases as x_0 becomes large.

Lemma 9. [7]

Consider the Riccati equation (4) and the assumptions of Theorem 3. Let $T^\star > 0$ be fixed. There exist two scalars $\beta_2 > 0$ and $\mu > 0$ such that

$$\bar{S}_k^{(+)} \leq \beta_2 Id$$

for all $T^\star \leq \bar{\tau}_k$, $k \in \mathbb{N}$, for all subdivisions $\left\{\bar{s}_k^{(i)}\right\}_{k\in\mathbb{N}}$, $\{\bar{\tau}_k\}_{k\in\mathbb{N}}$ such that $\bar{\tau}_k - \bar{\tau}_{k-1} < \mu$. This bound is also valid during prediction intervals.

A.2 Lower bound

We now prove that $\bar{S}(\bar{\tau})$ is also lower bounded for times greater than a fixed $T_\star > T^\star$.

Lemma 10. [7, 19]

Let $\bar{S} : [0, T[\to S_m$ (possibly with $T = +\infty$) be a solution of

$$\frac{d\bar{S}}{d\bar{\tau}} = -\mathcal{A}^{'}(\bar{\tau})\bar{S} - \bar{S}\mathcal{A}(\bar{\tau}) - \bar{S}Q\bar{S}$$

then, for any $\lambda \in \mathbb{R}^*$, for all $\bar{\tau} \in [0, T[$:

$$\begin{aligned}\bar{S}(\bar{\tau}) = e^{-\lambda\bar{\tau}}\varphi_a(\bar{\tau}, 0)\bar{S}_0\varphi_a^{'}(\bar{\tau}, 0)\\ +\lambda\int_0^{\bar{\tau}} e^{-\lambda(\bar{\tau}-v)}\varphi_a(\bar{\tau}, v)\left(\bar{S}(v) - \frac{\bar{S}(v)Q\bar{S}(v)}{\lambda}\right)\varphi_a^{'}(\bar{\tau}, v)dv\end{aligned} \tag{23}$$

where $\varphi_a(\bar{\tau}, s)$ is such that: $\left\{\begin{array}{rcl}\dfrac{d\varphi_a(\bar{\tau}, s)}{d\bar{\tau}} & = & -\mathcal{A}^{'}(\bar{\tau})\varphi_a(\bar{\tau}, s)\\ \varphi_a(s, s) & = & Id\end{array}\right.$

Lemma 11. [7, 19]

Let $\bar{S} : [0; e(\bar{S})[\to S_m$ be a maximal positive semi definite solution of

$$\frac{d}{d\bar{\tau}}\bar{S} = -\mathcal{A}^{'}\bar{S} - \bar{S}\mathcal{A} - \bar{S}Q\bar{S}$$

If $\bar{S}(0) = \bar{S}_0$ is positive definite then

$$e(\bar{S}) = +\infty \text{ and } \bar{S}(\bar{\tau}) \text{ is positive definite for all } \bar{\tau} \geq 0.$$

Remark 3. *As a consequence, the solution to the asynchronous continuous discrete Riccati equation (4) is positive definite for all time, and Lemma 9 is also valid for all times. In the rest of the present section, we show the existence of a lower bound of the form αId, for some positive scalar α.*

Following Lemma 10, and for a fixed $\lambda > 0$, $\bar{S}_1^{(+)}$ is written:

$$\begin{aligned}\bar{S}_1^{(+)} = e^{-\lambda\bar{\tau}_1}\varphi_a(\bar{\tau}_1, 0)\bar{S}_0\varphi_a^{'}(\bar{\tau}_1, 0)\\ +\lambda\int_0^{\bar{\tau}_1} e^{-\lambda(\bar{\tau}_1-v)}\varphi_a(\bar{\tau}_1, v)\left(\bar{S}(v) - \frac{\bar{S}(v)Q\bar{S}(v)}{\lambda}\right)\varphi_a^{'}(\bar{\tau}_1, v)dv\\ +\sum_{i\in\sigma_1} C^{(\mathfrak{s}_i)'}R^{(\mathfrak{s}_i)^{-1}}C^{(\mathfrak{s}_i)}\left(\bar{s}^{(i)}_{l_1^{(i)}} - \bar{s}^{(i)}_{l_1^{(i)}-1}\right)\end{aligned} \tag{24}$$

At time $\bar{\tau}_2$, the formula yields:

$$\begin{aligned}\bar{S}_2^{(+)} = e^{-\lambda(\bar{\tau}_2-\bar{\tau}_1)}\varphi_a(\bar{\tau}_2, \bar{\tau}_1)\bar{S}_1^{(+)}\varphi_a^{'}(\bar{\tau}_2, \bar{\tau}_1)\\ +\lambda\int_{\bar{\tau}_1}^{\bar{\tau}_2} e^{-\lambda(\bar{\tau}_2-v)}\varphi_a(\bar{\tau}_2, v)\left(\bar{S}(v) - \frac{\bar{S}(v)Q\bar{S}(v)}{\lambda}\right)\varphi_a^{'}(\bar{\tau}_2, v)dv\\ +\sum_{i\in\sigma_2} C^{(\mathfrak{s}_i)'}R^{(\mathfrak{s}_i)^{-1}}C^{(\mathfrak{s}_i)}\left(\bar{s}^{(i)}_{l_2^{(i)}} - \bar{s}^{(i)}_{l_2^{(i)}-1}\right)\end{aligned}$$

Replacing $\bar{S}_1^{(+)}$ by the expression obtained in (24) leads to:

$$\begin{aligned}\bar{S}_2^{(+)} &= e^{-\lambda\bar{\tau}_2}\varphi_a(\bar{\tau}_2,0)\bar{S}_0\varphi_a^{'}(\bar{\tau}_2,0)\\ &+\lambda\int_0^{\bar{\tau}_2} e^{-\lambda(\bar{\tau}_2-v)}\varphi_a(\bar{\tau}_2,v)\left(\bar{S}(v)-\frac{\bar{S}(v)Q\bar{S}(v)}{\lambda}\right)\varphi_a^{'}(\bar{\tau}_2,v)dv\\ &+e^{-\lambda(\bar{\tau}_2-\bar{\tau}_1)}\varphi_a(\bar{\tau}_2,\bar{\tau}_1)\sum_{i\in\sigma_1} C^{(\mathfrak{s}_i)'}R^{(\mathfrak{s}_i)^{-1}}C^{(\mathfrak{s}_i)}\left(\bar{s}^{(i)}_{l_1^{(i)}}-\bar{s}^{(i)}_{l_1^{(i)}-1}\right)\varphi_a^{'}(\bar{\tau}_2,\bar{\tau}_1)\\ &+\sum_{i\in\sigma_2} C^{(\mathfrak{s}_i)'}R^{(\mathfrak{s}_i)^{-1}}C^{(\mathfrak{s}_i)}\left(\bar{s}^{(i)}_{l_2^{(i)}}-\bar{s}^{(i)}_{l_2^{(i)}-1}\right)\end{aligned}$$

We iterate this procedure in order to compute $S_k(+)$ for any k:

$$\begin{aligned}\bar{S}_k^{(+)} &= e^{-\lambda\bar{\tau}_k}\varphi_a(\bar{\tau}_k,0)\bar{S}_0\varphi_a^{'}(\bar{\tau}_k,0)\\ &+\lambda\int_0^{\bar{\tau}_k} e^{-\lambda(\bar{\tau}_k-v)}\varphi_a(\bar{\tau}_k,v)\left(\bar{S}(v)-\frac{\bar{S}(v)Q\bar{S}(v)}{\lambda}\right)\varphi_a^{'}(\bar{\tau}_k,v)dv\\ &+\sum_{j=1}^{k}\sum_{i\in\sigma_j} e^{-\lambda(\bar{\tau}_k-\bar{\tau}_j)}\varphi_a(\bar{\tau}_k,\bar{\tau}_j)C^{(\mathfrak{s}_i)'}R^{(\mathfrak{s}_i)^{-1}}C^{(\mathfrak{s}_i)}\varphi_a^{'}(\bar{\tau}_k,\bar{\tau}_j)\left(\bar{s}^{(i)}_{l_j^{(i)}}-\bar{s}^{(i)}_{l_j^{(i)}-1}\right)\end{aligned} \tag{25}$$

This last equation is of the form $S_k(+) = (I) + (II) + (III)$, in this order.

(I) Since $\bar{S}_0$ is positive definite, (I) is at least positive semi-definite.

(II) Let us pick $\lambda > \beta\bar{q}$, then $\left(\bar{S}(v) - \frac{\bar{S}(v)Q\bar{S}(v)}{\lambda}\right)$ is positive definite, and (II) is at least positive semi-definite.

We now concentrate our efforts on (III) since it is the quantity that is actually bounded from below for all $\bar{\tau} > \bar{T}^\star$.

Let us define[8]:

1. the time $0 < \rho < \bar{\tau}_k$ such that $\bar{\tau}_k - \rho = T_\star$;
2. the index $\lambda_\rho^{(i)}$ as $\lambda_\rho^{(i)} = \max\{l \in \mathbb{N} : \bar{s}_l^{(i)} \leq \rho\}$ which always exists as soon as $\bar{\tau}_k > T_\star$.

Then, we use relation (22) that appears in the proof of lemma 7 to rewrite (III) as

$$(III) = \sum_{i=1}^{n_s}\sum_{j=1}^{\lambda_k^{(i)}} e^{-\lambda\left(\bar{\tau}_k-\bar{s}_j^{(i)}\right)}\varphi_a\left(\bar{\tau}_k,\bar{s}_j^{(i)}\right)C^{(\mathfrak{s}_i)'}R^{(\mathfrak{s}_i)^{-1}}C^{(\mathfrak{s}_i)}\varphi_a^{'}\left(\bar{\tau}_k,\bar{s}_j^{(i)}\right)\left(\bar{s}_j^{(i)}-\bar{s}_{j-1}^{(i)}\right)$$

[8] ρ is defined *w.r.t.* k —since we need our relations to remain valid for any $\bar{\tau}_k$ large enough— and should be understood as ρ_k. This latter notation is however not used for readability reasons.

Since all the terms of the sum (III) are symmetric positive semidefinite matrices:

$$(III) \geq \sum_{i=1}^{n_s} \sum_{j=\lambda_\rho^{(i)}+1}^{\lambda_k^{(i)}} e^{-\lambda\left(\bar{\tau}_k - \bar{s}_j^{(i)}\right)} \varphi_a\left(\bar{\tau}_k, \bar{s}_j^{(i)}\right) C^{(\mathfrak{s}_i)'} R^{(\mathfrak{s}_i)^{-1}} C^{(\mathfrak{s}_i)} \varphi_a'\left(\bar{\tau}_k, \bar{s}_j^{(i)}\right)\left(\bar{s}_j^{(i)} - \bar{s}_{j-1}^{(i)}\right)$$

From the properties of the resolvent φ_a, the above inequality can be rewritten, with $\bar{a}(\bar{\tau}) = a(\bar{\tau} + \rho)$:

$$(III) \geq \sum_{i=1}^{n_s} \sum_{j=\lambda_\rho^{(i)}+1}^{\lambda_k^{(i)}} e^{-\lambda\left(\bar{\tau}_k - \bar{s}_j^{(i)}\right)} \varphi_{\bar{a}}\left(\bar{\tau}_k - \rho, \bar{s}_j^{(i)} - \rho\right) C^{(\mathfrak{s}_i)'} R^{(\mathfrak{s}_i)^{-1}} C^{(\mathfrak{s}_i)} \varphi_{\bar{a}}'\left(\bar{\tau}_k - \rho, \bar{s}_j^{(i)} - \rho\right)\left(\bar{s}_j^{(i)} - \bar{s}_{j-1}^{(i)}\right)$$

If we denote by μ_i the maximum time step of a subdivision $\left\{s_k^i\right\}_{k\in\mathbb{N}}$, we notice that $e^{-\lambda\left(\bar{\tau}_k - \bar{s}_j^{(i)}\right)} \geq e^{-\lambda(T_\star + \mu_i)}$. Since $\bar{R}_{\sigma_k}^{-1}$ is defined in a compact subset, therefore, we need to find a lower bound for the following expression:

$$\sum_{i=1}^{n_s} \sum_{j=\gamma_k^{(i)}+1}^{\lambda_k^{(i)}} \varphi_{\bar{a}}\left(\bar{\tau}_k - \bar{s}_{\gamma_k^{(i)}}^{(i)}, \bar{s}_j^{(i)} - s_{\gamma_k^{(i)}}^{(i)}\right) C^{(\mathfrak{s}_i)'} C^{(\mathfrak{s}_i)} \varphi_{\bar{a}}'\left(\bar{\tau}_k - \bar{s}_{\gamma_k^{(i)}}^{(i)}, \bar{s}_j^{(i)} - \bar{s}_{\gamma_k^{(i)}}^{(i)}\right)\left(\bar{s}_j^{(i)} - \bar{s}_{j-1}^{(i)}\right)$$

Let us first redefine the subdivisions as follows:

- we denote $\hat{s}_j^{(i)} = \bar{s}_{j+\lambda_\rho^{(i)}}^{(i)} - \rho$, with $\hat{s}_0^{(i)} = 0$ for all $i \in \{1, ..., n_s\}$;
- each new subdivision $\{\hat{s}_j^{(i)}\}$ has $k_*^{(i)} + 1$ elements, with $k_*^{(i)} = \lambda_k^{(i)} - \lambda_\rho^{(i)}$. Hence, $\hat{s}_{k_*^{(i)}}^{(i)} = \bar{s}_{\lambda_k^{(i)}}^{(i)} - \rho$ for all $i \in \{1, ..., n_s\}$;
- we denote the subdivision $\{\hat{\tau}_j\}$ by $\{\hat{\tau}_j\} = \bigcup_i \left\{\hat{s}_j^{(i)}\right\}$, where elements belonging to several subdivisions are considered only once.

Thus, we can show that (III) has a lower bound if we can prove that

$$\sum_{i=1}^{n_s} \sum_{j=1}^{k_*^{(i)}} \varphi_{\bar{a}}\left(T_\star, \hat{s}_j^{(i)}\right) C^{(\mathfrak{s}_i)'} C^{(\mathfrak{s}_i)} \varphi_{\bar{a}}'\left(T_\star, \hat{s}_j^{(i)}\right)\left(\hat{s}_j^{(i)} - \hat{s}_{j-1}^{(i)}\right) \tag{26}$$

has a lower bound for all subdivisions $\{\hat{\tau}_j\}$ and $\left\{\hat{s}_j^{(i)}\right\}$, $i \in \{1, ..., n_s\}$, having a maximum time step size denoted by μ_i.

Let us now define $\psi_{\bar{a}}(\bar{\tau}, s) = \left(\varphi_{\bar{a}}^{-1}(\bar{\tau}, s)\right)'$, which is in fact the resolvent of system $\dot{x} = \mathcal{A}(\bar{\tau})x(\bar{\tau})$. Since $\psi_{\bar{a}}(\bar{\tau}, s) = \psi_{\bar{a}}^{-1}(s, \bar{\tau})$, we can rewrite (26) as follows:

$$G_{acd}(T_\star) = \sum_{i=1}^{n_s} \sum_{j=1}^{k_*^{(i)}} \psi_{\bar{a}}'\left(\hat{s}_j^{(i)}, T_\star\right) C^{(\mathfrak{s}_i)'} C^{(\mathfrak{s}_i)} \psi_{\bar{a}}\left(\hat{s}_j^{(i)}, T_\star\right)\left(\hat{s}_j^{(i)} - \hat{s}_{j-1}^{(i)}\right) \tag{27}$$

We call this latter quantity the *asynchronous continuous-discrete Gram observability matrix* associated to a time $T^\star > 0$. It is actually the key object that allows us to lower bound the Riccati matrix $\bar{S}_k$. In the following we show that, provided the time steps are small enough, $G_{acd}(T_\star)$ is as close as needed to the continuous time Gram observability matrix. To do so, we need the two following extra lemmas.

Lemma 12. e.g. [19]

Let $\Psi_a(\tau, s)$ denote the resolvent of the following time-dependent, observable, system:

$$\begin{aligned} \dot{x} &= \mathcal{A}(\tau)x(\tau) \\ y(\tau) &= Cx(\tau) \end{aligned}$$

where all the elements of the matrices $\mathcal{A}$ and C belongs to $L^\infty([0,T], \mathbb{R})$, and are uniformly bounded w.r.t. the L^∞ norm by some positive scalar $B > 0$. For a given $T > 0$, the (continuous) Gram observability matrix is defined as

$$G_c(T) = \int_0^T \psi_a^{'}(v,T) C^{'} C \psi_a(v,T) dv \tag{28}$$

Then, there exist positive scalars $0 < \mathfrak{a} < \mathfrak{b}$ depending on B and T only, such that

$$\mathfrak{a} Id \leq G_c(T) \leq \mathfrak{b} Id$$

Lemma 13.

Let $m(t)$, $t \in [0,T]$, be a $(n \times n)$ symmetric matrix, at least differentiable once.
Let μ be a positive constant, and $\{\bar{\tau}_j\}_{j\in\mathbb{N}}$ an arbitrary subdivision of $[0,T]$ such that $\bar{\tau}_j - \bar{\tau}_{j-1} \leq \mu$, for all $j \in \mathbb{N}$, with $\bar{\tau}_0 = 0$ and $\bar{\tau}_k$ the maximal element of the subdivision such that $T - \bar{\tau}_k \leq \mu$. We suppose that all the coefficients of m have their derivative bounded over time.
Then

$$\int_0^T m(v)dv - \sum_{j=1}^{k} m(\bar{\tau}_j)\left(\bar{\tau}_j - \bar{\tau}_{j-1}\right) \leq \mu\left(KT + L\right) Id$$

where $L = \sup_{\bar{\tau}} \|m(\bar{\tau})\|_2$, with $\|.\|_2$ the matrix norm induced by the euclidean norm, $K = \frac{n}{2} \max_{k,l,\bar{\tau}} \left(\left|m_{k,l}^{'}(\bar{\tau})\right|\right)$, with $m_{k,l}^{'}(\bar{\tau})$ the element of the kth row and l^{th} column of the matrix $m^{'}(\bar{\tau})$.

Proof. The proof of this lemma is mainly based on that of Lemma 3.11 in [7], with small differences discussed in Remark 4 at the end of the present section.

Let $M(t)$ be a primitive matrix of $m(t)$, that is to say a matrix whose elements are the primitives of the elements of $m(t)$. We have the identity

$$\int_0^T m(v)dv = M(T) - M(0) = \sum_{j=1}^{k} \left[M(\bar{\tau}_j) - M(\bar{\tau}_{j-1})\right] + \int_{\bar{\tau}_k}^T m(v)dv$$

We can apply the Taylor-Lagrange expansion on each element M_{kl}:

$$M_{kl}(\bar{\tau}_{i-1}) = M_{kl}(\bar{\tau}_i) + (\bar{\tau}_{i-1} - \bar{\tau}_i)\, m_{kl}(\bar{\tau}_i) + \frac{(\bar{\tau}_{i-1} - \bar{\tau}_i)^2}{2} m_{kl}^{'}(\xi_{kl,i})$$

where $\xi_{kl,i} \in [\bar{\tau}_{i-1}, \bar{\tau}_i]$. We have thus, the relation

$$\sum_{i=1}^{k} M(\bar{\tau}_{i-1}) = \sum_{i=1}^{k} M(\bar{\tau}_i) + \sum_{i=1}^{k} m(\bar{\tau}_i)\,(\bar{\tau}_{i-1} - \bar{\tau}_i) + \sum_{i=1}^{k} \left(\frac{(\bar{\tau}_{i-1} - \bar{\tau}_i)^2}{2} \mathfrak{R}_i \right)$$

where $(\mathfrak{R}_i)_{kl} = m_{kl}^{'}(\xi_{kl,i})$. Therefore

$$\begin{aligned}
\int_0^T m(v)dv - \sum_{i=1}^{k} m(\bar{\tau}_i)\,(\bar{\tau}_i - \bar{\tau}_{i-1}) &= \sum_{i=1}^{k} [M(\bar{\tau}_i) - M(\bar{\tau}_{i-1})] - \sum_{i=1}^{k} (\bar{\tau}_i - \bar{\tau}_{i-1}) m(\bar{\tau}_i) + \int_{\bar{\tau}_k}^T m(v)dv \\
&= \int_{\bar{\tau}_k}^T m(v)dv - \sum_{i=1}^{k} \left(\frac{(\bar{\tau}_{i-1} - \bar{\tau}_i)^2}{2} \mathfrak{R}_i \right) = \mathfrak{A} + \mathfrak{B}
\end{aligned}$$

We now use the definition of the matrix inequality to upper bound matrix $\mathfrak{B}$. Let x be a non zero element of $\mathbb{R}^n$:

$$\begin{aligned}
x^{'} \left[-\sum_{i=1}^{k} \left(\frac{(\bar{\tau}_{i-1} - \bar{\tau}_i)^2}{2} \mathfrak{R}_i \right) \right] x &= -\frac{1}{2} \sum_{i=1}^{k} \left((\bar{\tau}_{i-1} - \bar{\tau}_i)^2\, x^{'} \mathfrak{R}_i x \right) \\
&\leq \frac{1}{2} \sum_{i=1}^{k} \left((\bar{\tau}_{i-1} - \bar{\tau}_i)^2 \sum_{k,l} |x_k|\, |\mathfrak{R}_i|_{kl}\, |x_l| \right) \\
&\leq \frac{1}{2} \mu \max_{k,l,i} (|\mathfrak{R}_i|_{kl}) \left(\sum_{i=1}^{k} \bar{\tau}_{i-1} - \bar{\tau}_i \right) \left(\sum_{k,l} |x_k|\, |x_l| \right) \\
&\leq \frac{1}{2} \mu \max_{k,l,i} (|\mathfrak{R}_i|_{kl}) \left(\sum_{i=1}^{k} \bar{\tau}_{i-1} - \bar{\tau}_i \right) \frac{1}{2} \left(\sum_{k,l} |x_k|^2 + |x_l|^2 \right) \\
&\leq \mu \frac{n}{2} \max_{k,l,i} (|\mathfrak{R}_i|_{kl}) T \|x\|^2
\end{aligned}$$

Let us now upper bound matrix $\mathfrak{A}$. Since $m(\bar{\tau})$ is symmetric, for a given $\bar{\tau} \in \mathbb{R}^+$, $m(\bar{\tau}) \leq \|m(\bar{\tau})\|_2 Id$ where $\|.\|_2$ is the matrix norm induced by the euclidian norm, i.e. $\|m(\bar{\tau})\|_2 = \sup_{\|x\|_2} \|m(\bar{\tau})x\|_2$. Thus $\int_{\bar{\tau}_k}^T m(\bar{\tau}) \leq \sup_{\bar{\tau}} \|m(\bar{\tau})\|_2\, \mu Id$.

Those two upper bounds give us the result. □

The two preceding lemmas allow us to conclude this section's proof.

Lemma 14.

Consider the Riccati equation (4), and the assumptions of Theorem 3. Let $T_\star > T^\star$ be fixed. Then, there exist constants $\mu_i > 0$, $i \in \{1, ..., n_s\}$, and $\alpha_2 > 0$ such that, for all subdivisions $\left\{ \bar{s}_k^{(i)} \right\}_{k \in \mathbb{N}}$, $\{\bar{\tau}_k\}_{k \in \mathbb{N}}$ with $\left(\bar{s}_k^{(i)} - \bar{s}_{k-1}^{(i)} \right) \leq \mu_i$,

$$\alpha_2 Id \leq \bar{S}_k^{(+)} \quad \textit{as soon as} \quad \bar{\tau}_k > T_\star$$

Proof.

We start from Equation (27), the asynchronous continuous-discrete Gram observability matrix at time $T^\star > 0$:

$$G_{acd}(T_\star) = \sum_{i=1}^{n_s} G_{acd}^{(i)}(T_\star) = \sum_{i=1}^{n_s} \sum_{j=1}^{k_*^{(i)}} \psi_{\bar{a}}^{'}\left(\hat{s}_j^{(i)}, T_\star\right) C^{(\mathfrak{s}_i)'} C^{(\mathfrak{s}_i)} \psi_{\bar{a}}\left(\hat{s}_j^{(i)}, T_\star\right)\left(\hat{s}_j^{(i)} - \hat{s}_{j-1}^{(i)}\right)$$

Let us consider the continuous Gram matrix $G_c(T^\star)$, defined in Lemma 13, which also writes:

$$G_c(T_\star) = \sum_{i=1}^{n_s} \psi_a^{'}(v, T_\star) C^{(\mathfrak{s}_i)'} C^{(\mathfrak{s}_i)} \psi_a(v, T_\star) = \sum_{i=1}^{n_s} G_c^{(i)}(T_\star)$$

By lemma 13, for all $i \in \{1, ..., n_s\}$, there are constants $L > 0$ and $K_i > 0$:

$$G_c^{(i)}(T_\star) - G_{acd}^{(i)}(T_\star) \leq \mu_i(K_i T_\star + L) Id \tag{29}$$

Let us apply Lemma 12 on $G_c(T_\star)$:

$$\begin{aligned} \alpha Id &\leq G_c(T_\star) \\ &\leq \sum_{i=1}^{n_s} G_c^{(i)}(T_\star) - \sum_{i=1}^{n_s} G_{acd}^{(i)}(T_\star) + \sum_{i=1}^{n_s} G_{acd}^{(i)}(T_\star) \\ &\leq \sum_{i=1}^{n_s} G_{acd}^{(i)}(T_\star) + \sum_{i=1}^{n_s} \mu_i(K_i T_\star + L) Id \end{aligned}$$

Therefore

$$\left[\alpha - \sum_{i=1}^{n_s} \mu_i(K_i T_\star + L)\right] Id \leq G_{acd}(T_\star)$$

As a consequence, if all the μ_i are such that $\left(\alpha - \sum_{i=1}^{n_y} \mu_i(K_i T_\star + L)\right) > 0$, then, independently from the shape of the subdivisions $\left\{\hat{s}_k^{(i)}\right\}_{k\in\mathbb{N}}$ and $\{\hat{\tau}_k\}_{k\in\mathbb{N}}$, there exist a positive α_2 such that:

$$\alpha_2 Id \leq \bar{S}_k^{(+)}$$

This bound is also valid during prediction intervals. □

Remark 4. Erratum to [7]. *The reason why we need Lemma 13 instead of simply re-using Lemma 3.11 of [7] is because it should be used there as well. Indeed, the following mistake —which doesn't invalidate the main result of the article and is corrected by Lemma 13— is done in [7].*

The very end of Proposition 3.12, which corresponds to Lemma 14 in the present paper relies on the relation:

$$\mathfrak{a} Id \leq G_c(T^\star) \leq G_c(T^\star + \varepsilon) \quad \textit{for} \quad \varepsilon > 0 \tag{30}$$

Going back to the definition of the Gram observability matrix (28)*, we see that the argument of* $G_c(.)$ *plays a part both as the integration upper limit, but also in the definition of the resolvent matrix* ψ_a*. As such, the integrands of* $G_c(T^\star)$ *and* $G_c(T^\star + \varepsilon)$ *are not the same functions, which implies that 30 is not always true.*

However, this issue is resolved by following the procedure used in Equation (29) *above.*

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