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Abstract

We prove the existence of a weak solution to the compressible Navier–Stokes system with hard sphere possibly non-monotone pressure law involving, in particular, the Carnahan–Starling model [2] largely employed in various physical and industrial applications. We take into account large velocities prescribed at the boundary of a bounded piecewise C^2 domain and large densities prescribed at the inflow boundary without any restriction neither on the shape of the inflow/outflow boundaries nor on the shape of the domain.

Keywords: Compressible Navier–Stokes system, inhomogeneous boundary conditions, weak solutions, renormalized continuity equation, large inflow, large outflow, hard sphere pressure law

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1 Introduction

We consider the problem of identifying the non-steady motion of a compressible viscous fluid driven by the general in/out flux boundary conditions on general bounded domains. Specifically, the mass density $\varrho = \varrho(t,x)$ and the velocity $\mathbf{u} = \mathbf{u}(t,x)$, $(t,x) \in Q_T \equiv I \times \Omega$, $I = (0,T)$ of the fluid satisfy the Navier–Stokes system,

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}), \tag{1.2}$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} \right) + \lambda \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \mu > 0, \ \lambda \geq 0, \tag{1.3}$$

in $\Omega \subset R^d$, $d = 2,3$, where $p = p(\varrho)$ is the hard sphere pressure. The system is endowed with initial conditions

$$\varrho(0) = \varrho_0, \quad \varrho \mathbf{u}(0) = \varrho_0 \mathbf{u}_0. \tag{1.4}$$

We consider the general boundary conditions,

$$\mathbf{u}|_{\partial\Omega} = \mathbf{u}_B, \quad \varrho|_{\Gamma_{\rm in}} = \varrho_B, \tag{1.5}$$

where

$$\Gamma_{\rm in} = \left\{ x \in \partial\Omega \ \Big| \ \mathbf{u}_B \cdot \mathbf{n} < 0 \right\}, \quad \Gamma_{\rm out} = \left\{ x \in \partial\Omega \ \Big| \ \mathbf{u}_B \cdot \mathbf{n} > 0 \right\}. \tag{1.6}$$

In [3], it was established that a weak solution to the problem (1.1)–(1.6) with the barotropic pressure law (including isentropic pressure $p(\varrho) = a\varrho^\gamma$, $a > 0$) exists. The goal of this paper is to establish the existence of a weak solution $(\varrho, \mathbf{u})$ to the problem (1.1)–(1.6) for general *large* boundary data ϱ_B and $\mathbf{u}_B$ under the following physically grounded hypothesis (see Carnahan and Starling [2] among many others):

- **Molecular hypothesis (hard sphere model).** The specific volume of the fluid is bounded below away from zero. Equivalently, the fluid density cannot exceed a limit value $\overline{\varrho} > 0$. Accordingly, the pressure $p = p(\varrho)$ satisfies

$$\lim_{\varrho \to \overline{\varrho}} p(\varrho) = \infty.$$

Although apparently satisfied by any *real* fluid, this condition eliminates the more standard equations of state used for the isentropic gases.

Since we focus on the inflow/outflow phenomenon, we deliberately omit the contribution of external forces $\varrho \mathbf{f}$. Nevertheless, all the results of this paper are valid even when there are external forces.

It is important to investigate the equations in this setting and to get better insight in many real-world applications. In fact, this is a natural and basic abstract setting for flows in some specific examples such as pipelines, wind tunnels, and turbines. In spite of this fact, the literature on this problem is in a short supply. To the best of our knowledge, this is the first work treating this system with the hard sphere pressure law for large boundary data in a very large class of bounded domains.

The most of available theories of weak solutions on arbitrary large time interval $(0,T)$ deals with the above system endowed with zero outflow and inflow boundary conditions, and with the barotropic pressure law: we mention monographs [17], [20], [6], [7] and references quoted there. Monographs [17] and [20] treat in the same context also the stationary problem. The case of fluid flow in the barotropic regime with large general inflow/outflow boundary conditions without restrictions on the shape of the boundary received an exhausting answer only recently in [3]. Papers by Novo [19] and by Girinon [15] deal with the same problem as [3], but there are severe restrictions on the boundary and the boundary data. The existence of strong solutions for the problem (1.1)–(1.6) on a short time interval and/or with small boundary data are better investigated since several decades, see e.g. Valli, Zajaczkowski [25] among others. Existence of weak stationary solutions in the barotropic regime is still open. The only available results in the steady regimes are those with small boundary data, see Plotnikov, Ruban, Sokolowski [23], [24], Mucha, Piasecki [18], Piasecki [21], Piasecki and Pokorny [22] among others.

The results about the existence of weak solutions for the hard sphere model are in a short supply. The existence of weak solutions for the problem without inflow/outflow is investigated in Feireisl, Zhang [10] and Feireisl, Lu, Málek [11]. The same problem with arbitrary large outflow and inflow data is so far open: its solution is the subject of the present paper. The stationary solutions in this situation have been constructed only recently in [8].

Clearly, the fact that the density is *a priori* expected to be confined to a bounded interval $[0,\overline{\varrho})$ facilitates the formal analysis. On the other hand, the presence of non-zero boundary data makes the analysis more difficult. The rigorous proof of the confinement of the density to the interval $[0,\overline{\varrho})$ as well as of the uniform bound and equi-integrability of the pressure are far to be obvious. Once the latter property is proved, one may employ the standard procedure of compactness for these equations involving effective viscous flux identity and DiPerna Lions transport theory [5] modified in [3] in order to accommodate the non-homogenous boundary conditions.

The paper is organized as follows. The main results are announced in Theorem 2.4 and Theorem 2.5 in Section 2. Theorem 2.4 provides the local in time existence of weak solutions while Theorem 2.5 provides global-in-time result. Both theorems hold under the condition on the existence of a convenient extension of the boundary velocity field. The condition for local-in-time existence is less severe than that for global existence. Both theorems are proved in Sections 3–4. The singular pressure is set to be constant at distance $\varepsilon > 0$ left from $\overline{\varrho}$ and extended through the barotropic law with a large adiabatic coefficient at a positive distance right from $\overline{\varrho}$. The approximate system (with large non-homogenous boundary data and the above regularized barotropic pressure) is introduced in Section 3. It benefits from an existence theorem due to [3], and as such admits a weak solution. The existence of a weak solution and its properties are recalled in Theorem 3.1. Uniform estimates for the sequence of weak solutions are derived in Section 4.1; they allow to pass to the limit in the continuity equation and to show the boundedness of density in Section 4.2. Uniform integrability of the pressure sequence is derived in Section 4.3. Once this is known, the equi-integrability of the pressure is derived in Section 4.4. Section 4.5 deals with the limit in the momentum equation and Section 4.6 is devoted to the effective viscous flux identity, while Section 4.7 concludes the proof by evaluating the oscillations in the density sequence by using the renormalized continuity equation.

This paper provides a quite complete picture on the conditions for local/ global-in-time existence of weak solutions in terms of the total velocity flux $\int_{\partial\Omega} \mathbf{u}_B \cdot \mathbf{n} \mathrm{d}S_x$ through the boundary. Loosely speaking, if the total velocity flux through the boundary is positive, then there is a global-in-time weak solution, while if it is negative, then a weak solution may fail to exist on an arbitrarily large time interval. If the total velocity flux through the boundary is zero, there is always at least a local in time weak solution. In Section 5, these problems are investigated and corresponding underlying extensions are also constructed.

The existence of (even) local-in-time weak solutions in the case

$$\int_{\partial\Omega} \mathbf{u}_B \cdot \mathbf{n} \mathrm{d}S_x < 0$$

and the existence of global-in-time weak solutions in the case

$$\int_{\partial\Omega} \mathbf{u}_B \cdot \mathbf{n} \mathrm{d}S_x = 0$$

are eminent open problems. In the latter case, there is only a positive answer in a “trivial” situation, $\mathbf{u}_B = 0$ (cf. [10] and [11]).

2 Main result

We suppose, for the sake of simplicity, that the boundary data satisfy

$$\varrho_B \in C(\partial\Omega), \quad \mathbf{u}_B \in C^2(\partial\Omega; R^3), \tag{2.1}$$

where

$$\Omega \in R^d,\ d = 2, 3 \text{ is a bounded domain of class } C^2. \tag{2.2}$$

In agreement with what was mentioned in the introduction, we assume that

$$p = \mathscr{p} - \mathfrak{p}$$

where $\mathscr{p} \in C[0, \overline{\varrho}) \cap C^1(0, \overline{\varrho})$ satisfies

$$\mathscr{p}(0) = 0, \quad \mathscr{p}' > 0, \quad \mathscr{p}(\varrho) \sim_{\varrho \to \overline{\varrho}-} |\overline{\varrho} - \varrho|^{-\beta} \text{ for some } \beta > 5/2 \tag{2.3}$$

and

$$\mathfrak{p} \in C^2_c[0, \underline{\varrho}), \quad \mathfrak{p} \geq 0, \quad \mathfrak{p}(0) = \mathfrak{p}'(0) = 0, \quad \text{where } 0 < \underline{\varrho} < \overline{\varrho}. \tag{2.4}$$

In the above $a(s) \sim_{s \to s_0 \pm} b(s)$ means that $c_1 a(s) \leq b(s) \leq c_2(s)$ in a right(+), left (−) neighborhood of s_0. Moreover, we may suppose $1 < \underline{\varrho} < \overline{\varrho} < \infty$ without loss of generality.

We consider the general non-monotone pressure. The usual monotone hard sphere pressure law is a particular case when the component $\mathfrak{p}$ of the pressure is identically zero.

We begin with the definition of weak solutions to system (1.1)–(1.6) with (2.3)–(2.4).

Definition 2.1 *We say that* $(\varrho, \mathbf{u})$ *is a bounded energy weak solution of the problem* (1.1)–(2.4) *on a time interval* $(0, T)$ *if the following four conditions are satisfied.*

1. *It belongs to the following functional spaces:*

$$0 \le \varrho < \overline{\varrho} \text{ a.a. in } (0,T)\times\Omega, \quad p(\varrho) \in L^1(0,T; L^1_{\rm loc}(\Omega)) \qquad (2.5)$$
$$\mathbf{u} \in L^2(0,T; W^{1,2}(\Omega; R^3)), \quad \mathbf{u}|_{I\times\partial\Omega} = \mathbf{u}_B.$$

2. *The function* $\varrho \in C_{\rm weak}([0,T], L^1(\Omega))$[1] *satisfies the integral identity*

$$\int_\Omega \varrho(\tau,\cdot)\varphi(\tau,\cdot)\ {\rm d}x - \int_\Omega \varrho_0(\cdot)\varphi(0,\cdot)\ {\rm d}x$$
$$= \int_0^\tau \int_\Omega \Big(\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi\Big)\ {\rm d}x{\rm d}t - \int_0^\tau\int_{\Gamma_{\rm in}} \varrho_B\mathbf{u}_B\cdot\mathbf{n}\varphi\ {\rm d}S_x{\rm d}t \qquad (2.6)$$

for any $\tau\in[0,T]$ *and* $\varphi \in C^1_c([0,T]\times(\Omega\cup\Gamma_{\rm in}))$.

3. *The function* $\varrho\mathbf{u} \in C_{\rm weak}([0,T], L^1(\Omega; R^3))$ *satisfies the integral identity*

$$\int_\Omega \varrho\mathbf{u}(\tau,\cdot)\cdot\boldsymbol{\varphi}(\tau,\cdot)\ {\rm d}x - \int_\Omega \varrho_0\mathbf{u}_0(\cdot)\cdot\boldsymbol{\varphi}(0,\cdot)\ {\rm d}x$$
$$= \int_0^\tau\int_\Omega \Big(\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\boldsymbol{\varphi} + p(\varrho){\rm div}_x\boldsymbol{\varphi} - \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\boldsymbol{\varphi}\Big)\ {\rm d}x{\rm d}t \qquad (2.7)$$

for any $\tau\in[0,T]$ *and any* $\boldsymbol{\varphi}\in C^1_c([0,T]\times\Omega; R^3)$.

4. *There is a Lipschitz extension* $\mathbf{u}_\infty \in W^{1,\infty}(\Omega)$ *of the vector field* $\mathbf{u}_B$ *such that the following energy inequality holds*

$$\int_\Omega \Big(\frac{1}{2}\varrho|\mathbf{u}-\mathbf{u}_\infty|^2 + H(\varrho)\Big)(\tau)\ {\rm d}x + \int_0^\tau\int_\Omega \mathbb{S}(\nabla_x(\mathbf{u}-\mathbf{u}_\infty)) : \nabla_x(\mathbf{u}-\mathbf{u}_\infty)\ {\rm d}x{\rm d}t$$
$$+ \int_0^\tau\int_\Omega p^-(\varrho){\rm div}\mathbf{u}_\infty\ {\rm d}x{\rm d}t + \int_0^\tau\int_K p^+(\varrho){\rm div}\mathbf{u}_\infty{\rm d}x{\rm d}t$$
$$\le \int_\Omega\Big(\frac{1}{2}\varrho_0|\mathbf{u}_0-\mathbf{u}_\infty|^2 + H(\varrho_0)\Big)(\tau)\ {\rm d}x - \int_0^\tau\int_\Omega \varrho\mathbf{u}\cdot\nabla_x\mathbf{u}_\infty\cdot(\mathbf{u}-\mathbf{u}_\infty)\ {\rm d}x{\rm d}t \qquad (2.8)$$
$$- \int_0^\tau\int_\Omega \mathbb{S}(\nabla_x\mathbf{u}_\infty) : \nabla_x(\mathbf{u}-\mathbf{u}_\infty)\ {\rm d}x{\rm d}t - \int_0^\tau\int_{\Gamma_{\rm in}} H(\varrho_B)\mathbf{u}_B\cdot\mathbf{n}{\rm d}S_x{\rm d}t$$
$$- \underline{H}\int_0^\tau\int_{\Gamma_{\rm out}} \mathbf{u}_B\cdot\mathbf{n}{\rm d}S_x{\rm d}t$$

for a.a. $\tau\in(0,T)$ *with any compact* $K\subset\Omega$, *where*

$$p^-(\varrho) = \min\{p(\varrho),0\}, \quad p^+(\varrho) = \max\{p(\varrho),0\}$$
$$H(\varrho) = \varrho\int_1^\varrho \frac{p(z)}{z^2}{\rm d}z, \quad \underline{H} := \inf_{\varrho>0} H(\varrho). \qquad (2.9)$$

---

[1] We say that $f\in C_{\rm weak}([0,T], L^p(\Omega))$ iff $\int_\Omega f\varphi\ {\rm d}x \in C[0,T]$ for all $\varphi\in L^{p'}(\Omega)$.

Remark 2.1. *1.* A brief inspection of (2.9) gives the estimate of value $\underline{H}$,

$$\underline{H} \geq - \sup_{\varrho\in(0,1)} \not{p}(\varrho) - \overline{\varrho} \sup_{\varrho>1} \mathfrak{p}(\varrho) > -\infty$$

provided $\mathrm{supp}\mathfrak{p} \subset [0,\underline{\varrho}]$. Likewise,

$$\inf_{\varrho>0} p^{-}(\varrho) \geq \underline{p} \geq - \sup_{\varrho>0} \mathfrak{p}(\varrho) > -\infty.$$

2. The continuity equation (2.6) yields the total mass inequality

$$\int_\Omega \varrho(\tau) \ \mathrm{d}x \leq \int_\Omega \varrho_0 \ \mathrm{d}x - \int_0^\tau \int_{\Gamma_{\rm in}} \varrho_B \mathbf{u}_B \cdot \mathbf{n} \mathrm{d}S_x \mathrm{d}t. \tag{2.10}$$

for all $\tau\in[0,T]$. It can be obtained by taking in (2.6) test functions $\varphi=\varphi_\varepsilon$ as in (4.47) and then by letting $\varepsilon\to 0$.

Definition 2.2 *We say that a couple* $(\varrho,\mathbf{u})$ *satisfying* $0\leq\varrho\leq\overline{\varrho}<\infty$ *and* $\mathbf{u}\in L^2(0,T;W^{1,2}(\Omega,R^3))$ *is a renormalized solution of the continuity equation if for any* $b\in C^1[0,\overline{\varrho}]$*, functions* ϱ *and* $b(\varrho)\in C_{\rm weak}([0,T];L^1(\Omega))$*, and it satisfies, in addition to the weak formulation of the continuity equation* (2.6)*, also the weak formulation of the renormalized equation,*

$$\begin{aligned} &\int_\Omega (b(\varrho)\mathbf{u})(\tau,\cdot)\varphi(\tau,\cdot) \ \mathrm{d}x - \int_\Omega (b(\varrho_0)\mathbf{u}_0)\varphi(0,\cdot) \ \mathrm{d}x \\ &= \int_0^\tau \int_\Omega \Big(b(\varrho)\partial_t\varphi + b(\varrho)\mathbf{u}\cdot\nabla_x\varphi - \varphi\left(b'(\varrho)\varrho - b(\varrho)\right)\mathrm{div}_x\mathbf{u}\Big) \ \mathrm{d}x\mathrm{d}t \\ &\quad - \int_0^\tau \int_{\Gamma_{\rm in}} b(\varrho_B)\mathbf{u}_B\cdot\mathbf{n}\varphi \ \mathrm{d}S_x\mathrm{d}t \end{aligned} \tag{2.11}$$

for any $\varphi\in C^1([0,T]\times(\Omega\cup\Gamma_{\rm in}))$*. A weak solution to the problem* (1.1)–(1.6) *satisfying in addition renormalized continuity equation* (2.11) *is called a renormalized weak solution.*

Remark 2.2. *1.* In fact any couple $(\varrho,\mathbf{u})$ in class (2.5) veryfying continuity equation (2.6) verifies also the renormalized continuity equation (2.11). This follows from the DiPerna–Lions transport theory [5] reshaped for the case of non-homogenous boundary conditions in [3, Lemma 4.4]. This lemma adapted to the present situation states:

Lemma 2.3. *Suppose that* $\Omega\subset R^3$ *is a bounded Lipschitz domain and* $(\varrho_B,\mathbf{u}_B)$ *satisfies the assumptions* (2.1)*. Assume further that the inflow portion* $\Gamma_{\rm in}$ *of the boundary is a* C^2 *open* $(d-1)$*-dimensional manifold. Suppose further that the couple* $(\varrho,\mathbf{u})\in L^2((0,T)\times\Omega;[0,\overline{\varrho}])\times L^2(0,T;W^{1,2}(\Omega;R^3))$ *satisfies the continuity equation in the weak sense (4.12). Then* $(\varrho,\mathbf{u})$ *is also a renormalized solution of the continuity equation (4.12), meaning that it verifies equation* (2.11) *for any* $\varphi\in C^1([0,T]\times(\Omega\cup\Gamma_{\rm in}))$ *and* $b\in C^1[0,\overline{\varrho}]$.

We are now in a position to announce the main results of this paper. The first result is a local-in-time existence theorem for weak solutions. It holds for the general boundary data with a restriction requiring non-negative total velocity flux over the boundary. It gives a lower bound of the maximal existence time of weak solutions in terms of the size of initial and boundary data.

Theorem 2.4. *Let $\Omega \subset R^d$, $d = 2, 3$, be a bounded domain of class C^2. Assume that the pressure satisfies the hypotheses* (2.3)–(2.4)*, the initial data have finite energy*

$$\mathscr{E}_0 := \int_\Omega \frac{1}{2}\varrho_0 \mathbf{u}_0^2 + H(\varrho_0)\ \mathrm{d}x < \infty, \quad \mathscr{M}_0 := \int_\Omega \varrho_0\ \mathrm{d}x > 0 \tag{2.12}$$

and the boundary data $\mathbf{u}_B$ and ϱ_B satisfy (2.1)*. Assume moreover that there exists $\mathbf{u}_\infty \in W^{1,\infty}(\Omega; R^3)$ such that*

$$\mathbf{u}_\infty|_{\partial\Omega} = \mathbf{u}_B, \quad \mathrm{div}\mathbf{u}_\infty \geq 0 \quad \textit{in } \Omega. \tag{2.13}$$

Finally, suppose that

$$0 < \underline{\varrho}_B \equiv \min_{\partial\Omega} \varrho_B \leq \max_{\partial\Omega} \varrho_B < \overline{\varrho}, \quad 0 \leq \inf_\Omega \varrho_0 \leq \sup_\Omega \varrho_0 < \overline{\varrho}. \tag{2.14}$$

Then there exists

$$T = T_{\max} \geq \frac{\overline{\varrho}|\Omega| - \int_\Omega \varrho_0\ \mathrm{d}x}{\int_{\Gamma_{\rm in}} \varrho_B |\mathbf{u}_B \cdot \mathbf{n}| \mathrm{d}S_x} \tag{2.15}$$

such that the problem (1.1)–(1.6) *admits at least one renormalized bounded energy weak solution $(\varrho, \mathbf{u})$ on $(0, T)$.*

The second result is a global-in-time existence theorem for weak solutions. It requires an additional restriction on the boundary velocity (which may be seen as a sign condition on the total boundary velocity flux through the boundary).

Theorem 2.5. *Suppose that the domain Ω, the pressure p, initial data $(\varrho_0, \mathbf{u}_0)$ and boundary data $(\varrho_B, \mathbf{u}_B)$ satisfy all hypotheses of Theorem 2.4 and that $T > 0$. Assume moreover that an extension $\mathbf{u}_\infty$ of $\mathbf{u}_B$ satisfies, in addition to* (2.13),

$$\mathrm{essinf}_{\mathscr{O}}\Big(\mathrm{div}\mathbf{u}_\infty\Big) \geq \underline{d} > 0 \quad \textit{where } \mathscr{O} \textit{ is an open set satisfying } \overline{\mathscr{O}} \subset \Omega. \tag{2.16}$$

Then the problem (1.1)–(1.6) *admits at least one renormalized bounded energy weak solution $(\varrho, \mathbf{u})$ on $(0, T)$.*

We shall perform the proofs of Theorem 2.4 and Theorem 2.5 in every detail for the case $d = 3$. The proofs for the case $d = 2$ are left to the reader as exercises. We end this section by giving a few remarks.

Remark 2.6. *1.* Theorem 2.4 and Theorem 2.5 still hold provided one considers in the momentum equation at its right-hand side term $\varrho\mathbf{f}$ corresponding to large external forces $\mathbf{f} \in L^\infty(Q_T)$ (modulo necessary changes in the weak formulation in order to accommodate the presence of this term). It is remarkable that the lower bound for the maximal existence time in the case of the local existence theorem is independent of the size of the external force.

2. None of the conditions (2.13) and (2.16) seems to be a necessary compatibility condition to guarantee the (local) global-in-time existence of weak solutions to the problem. Indeed, in the case of barotropic pressure, the global existence of weak solutions holds with arbitrary sufficiently smooth boundary velocity, see [3]. The main reason of this situation is the fact, that the Helmholtz function appearing in the energy inequality and providing in both cases the density estimate, is, in the barotropic case, comparable with the pressure, while, in the hard sphere model case, it is dominated by the pressure. Conditions (2.13) and (2.16) in Theorems 2.4, 2.5 are imposed in order to ran-over this difficulty and to enforce the density estimates.

In spite of this complication, the theory presented in this paper offers a quite exhausting picture on the solvability of problem (1.1)–(1.6) with (2.3)–(2.4) in terms of the sign condition of the total boundary velocity fluxes:

- Consider a sufficiently smooth simply connected domain and suppose that boundary velocity verifies

$$\int_{\partial\Omega} \mathbf{u}_B \cdot \mathbf{n} \mathrm{d}S_x = 0 \quad \text{or equivalently} \quad \int_{\Gamma_{\rm in}} |\mathbf{u}_B \cdot \mathbf{n}| \mathrm{d}S_x = \int_{\Gamma_{\rm out}} |\mathbf{u}_B \cdot \mathbf{n}| \mathrm{d}S_x.$$

  Then according to Lemma 5.1, there exists a solenoidal Lipschitz extension $\mathbf{u}_\infty$ of $\mathbf{u}_B$ (i.e., in particular, verifying condition (2.13)). In this case, Theorem 2.4 ensures local in time existence of bounded energy weak solutions without any further restriction on the boundary data.

  The problem on the existence of a global-in-time weak solution in this situation remains open in general. The only result is available in the simplest particular case when $\mathbf{u}_B \cdot \mathbf{n} = 0$ pointwise in $\partial\Omega$, see Feireisl, Zhang [10] or Feireisl, Lu, Málek [11].

- On a sufficiently smooth simply connected domain, both conditions (2.13) and (2.16) are satisfied provided the boundary velocity verifies

$$\int_{\partial\Omega} \mathbf{u}_B \cdot \mathbf{n} \mathrm{d}S_x > 0 \quad \text{or equivalently} \quad \int_{\Gamma_{\rm in}} |\mathbf{u}_B \cdot \mathbf{n}| \mathrm{d}S_x < \int_{\Gamma_{\rm out}} |\mathbf{u}_B \cdot \mathbf{n}| \mathrm{d}S_x.$$

  We refer the reader to consult Lemma 5.2 later. In this situation, the problem admits always a global-in-time bounded energy renormalized weak solution.

- If

$$\int_{\partial\Omega} \mathbf{u}_B \cdot \mathbf{n} \mathrm{d}S_x < 0 \quad \text{or equivalently} \quad \int_{\Gamma_{\rm in}} |\mathbf{u}_B \cdot \mathbf{n}| \mathrm{d}S_x > \int_{\Gamma_{\rm out}} |\mathbf{u}_B \cdot \mathbf{n}| \mathrm{d}S_x,$$

then the problem fails to have a global-in-time weak solution for any ϱ_B subjected to the constraints imposed by Theorem 2.4 regardless of the fact that the domain is simply connected or regular. An example of such situation is constructed in Lemma 5.4 for any bounded Lipschitz domain. Local existence of weak solutions in this situation remains an open problem.

3. If we can guarantee $p(\varrho) \in L^1(Q_T)$, or if either p or p^+ are convex on $[0,\infty)$ then any weak solution constructed in Theorem 2.4 and Theorem 2.5 verifies slightly stronger form of energy inequality than (2.8), where the contribution of the pressure at the left-hand side of (2.8) $\int_0^\tau\int_\Omega p^-(\varrho)\mathrm{div}\mathbf{u}_\infty\ \mathrm{d}x\mathrm{d}t + \int_0^\tau\int_K p^+(\varrho)\mathrm{div}\mathbf{u}_\infty\mathrm{d}x\mathrm{d}t$ is replaced by $\int_0^\tau\int_\Omega p(\varrho)\mathrm{div}\mathbf{u}_\infty\ \mathrm{d}x\mathrm{d}t$.

4. Theorem 2.4 and Theorem 2.5 remain valid on piecewise C^2 bounded domains. This generalization will be discussed in the last section.

3 Approximate problem

In order to construct the solutions in Theorem 2.4 and Theorem 2.5, we begin by considering the system (1.1)–(1.6) with p_ϵ instead of p where

$$p_\varepsilon = \mathcal{p}_\varepsilon - \mathfrak{p}, \tag{3.1}$$

and

$$\mathcal{p}_\varepsilon = \begin{cases} \mathcal{p}(\varrho) & \text{if } \varrho \in [0, \overline{\varrho} - \varepsilon] \\ \mathcal{p}(\overline{\varrho} - \varepsilon) + |(\varrho - \overline{\varrho} + \varepsilon)^+|^\gamma & \text{if } \varrho \in (\overline{\varrho} - \varepsilon, \infty) \end{cases}$$

with $\gamma > d$ (which must be chosen sufficiently large). In fact, for any $\beta > 2$ there exists $\gamma_0 > d/2$ such that for all $\gamma > \gamma_0$ (3.1) represents a convenient approximation. We noticed that $\gamma_0 \to \infty$ as $\beta \to 2+$. The choice of γ_0 as a function of β and the constraint $\beta > 2$ are dictated by the requirement to have sufficient estimates. The most restricting condition is to guaranteeing equi-integrability of the pressure, see (4.29). We shall suppose without loss of generality $0 < \varepsilon < (\overline{\varrho} - \underline{\varrho})/2$.

According to Theorem 2.3 and Remark 2.5 in [3], the system (1.1)–$(1.6)_{p=p_\varepsilon}$ admits at least one bounded energy weak solution. More precisely, the following theorem holds.

Theorem 3.1. *Let $\Omega \subset R^d$, $d = 2, 3$, be a bounded domain of class C^2 and $\gamma > d/2$. Let the boundary data $\mathbf{u}_B$ and ϱ_B satisfy* (2.1), *where* $\min \varrho_B > 0$. *Assume that the initial data are of finite energy* (2.12). *Then the problem* (1.1)–$(1.6)_{p=p_\varepsilon}$ *possesses at least one renormalized bounded energy weak solution* $(\varrho_\varepsilon, \mathbf{u}_\varepsilon)$, *i.e.*

1. *The couple $(\varrho_\varepsilon, \mathbf{u}_\varepsilon)$ belongs to the following functional space:*

$$\begin{aligned} &\varrho_\varepsilon \in L^\infty(0,T; L^\gamma(\Omega)), \quad 0 \le \varrho_\varepsilon \ \ a.a.\ in\ (0,T)\times\Omega, \\ &\mathbf{u}_\varepsilon \in L^2(0,T; W^{1,2}(\Omega; R^d)), \quad \mathbf{u}_\varepsilon|_{I\times\partial\Omega} = \mathbf{u}_B. \end{aligned} \tag{3.2}$$

2. *The function* $\varrho_\varepsilon \in C_{\rm weak}([0,T], L^\gamma(\Omega))$ *satisfies the integral identity*

$$
\begin{aligned}
&\int_\Omega \varrho_\varepsilon(\tau,\cdot)\varphi(\tau,\cdot)\ {\rm d}x - \int_\Omega \varrho_0(\cdot)\varphi(0,\cdot)\ {\rm d}x \\
&= \int_0^\tau \int_\Omega \Big(\varrho_\varepsilon \partial_t \varphi + \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \varphi\Big)\ {\rm d}x{\rm d}t - \int_0^\tau \int_{\Gamma_{\rm in}} \varrho_B \mathbf{u}_B \cdot \mathbf{n}\varphi\ {\rm d}S_x{\rm d}t
\end{aligned}
\tag{3.3}
$$

for any $\tau \in [0,T]$ *and* $\varphi \in C^1_c([0,T]\times(\Omega\cup\Gamma_{\rm in}))$*. In particular,*

$$
\int_\Omega \varrho_\varepsilon(\tau)\ {\rm d}x \leq \int_\Omega \varrho_0\ {\rm d}x - \int_0^\tau \int_{\Gamma_{\rm in}} \varrho_B \mathbf{u}_B \cdot \mathbf{n}{\rm d}S_x{\rm d}t. \tag{3.4}
$$

3. *The renormalized continuity equation*

$$
\begin{aligned}
&\int_\Omega b(\varrho_\varepsilon)(\tau,\cdot)\varphi(\tau,\cdot)\ {\rm d}x - \int_\Omega b(\varrho_0)(\cdot)\varphi(0,\cdot)\ {\rm d}x \\
&= \int_0^\tau \int_\Omega \Big(b(\varrho_\varepsilon)\partial_t\varphi + b(\varrho_\varepsilon)\mathbf{u}_\varepsilon\cdot\nabla_x\varphi + (b(\varrho_\varepsilon) - b'(\varrho_\varepsilon)\varrho_\varepsilon){\rm div}\mathbf{u}_\varepsilon\Big)\ {\rm d}x{\rm d}t \\
&\quad - \int_0^\tau \int_{\Gamma_{\rm in}} b(\varrho_B)\mathbf{u}_B\cdot\mathbf{n}\varphi\ {\rm d}S_x{\rm d}t
\end{aligned}
\tag{3.5}
$$

holds for any $b \in C[0,\infty)$ *with* $b' \in C_c[0,\infty)$*,* $\tau \in [0,T]$*, and* $\varphi \in C^1_c([0,T]\times(\Omega\cup\Gamma_{\rm in}))$*.*

4. *The function* $\varrho_\varepsilon\mathbf{u}_\varepsilon \in C_{\rm weak}([0,T], L^{\frac{2\gamma}{\gamma+1}}(\Omega;R^d))$ *satisfies the integral identity*

$$
\begin{aligned}
&\int_\Omega \varrho_\varepsilon\mathbf{u}_\varepsilon(\tau,\cdot)\cdot\boldsymbol{\varphi}(\tau,\cdot)\ {\rm d}x - \int_\Omega \varrho_0\mathbf{u}_0(\cdot)\boldsymbol{\varphi}(0,\cdot)\ {\rm d}x \\
&= \int_0^\tau \int_\Omega \Big(\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\partial_t\varphi + \varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon : \nabla_x\boldsymbol{\varphi} + p_\varepsilon(\varrho_\varepsilon){\rm div}_x\boldsymbol{\varphi} - \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\boldsymbol{\varphi}\Big)\ {\rm d}x{\rm d}t
\end{aligned}
\tag{3.6}
$$

for any $\tau \in [0,T]$ *and* $\boldsymbol{\varphi} \in C^1_c([0,T]\times\Omega;R^d)$*.*

5. *The energy inequality*

$$
\begin{aligned}
&\int_\Omega \Big(\frac{1}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon - \mathbf{u}_\infty|^2 + H_\varepsilon(\varrho_\varepsilon)\Big)(\tau)\ {\rm d}x + \int_0^\tau\int_\Omega \mathbb{S}(\nabla_x(\mathbf{u}_\varepsilon - \mathbf{u}_\infty)) : \nabla_x(\mathbf{u}_\varepsilon - \mathbf{u}_\infty)\ {\rm d}x{\rm d}t \\
&\leq \int_\Omega \Big(\frac{1}{2}\varrho_0|\mathbf{u}_0 - \mathbf{u}_\infty|^2 + H_\varepsilon(\varrho_0)\Big)\ {\rm d}x - \int_0^\tau\int_\Omega p_\varepsilon(\varrho_\varepsilon){\rm div}\mathbf{u}_\infty\ {\rm d}x{\rm d}t \\
&\quad - \int_0^\tau\int_\Omega \varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\nabla_x\mathbf{u}_\infty\cdot(\mathbf{u}_\varepsilon - \mathbf{u}_\infty)\ {\rm d}x{\rm d}t - \int_0^\tau\int_\Omega \mathbb{S}(\nabla_x\mathbf{u}_\infty) : \nabla_x(\mathbf{u}_\varepsilon - \mathbf{u}_\infty)\ {\rm d}x{\rm d}t \\
&\quad - \int_0^\tau\int_{\Gamma_{\rm in}} H_\varepsilon(\varrho_B)\mathbf{u}_B\cdot\mathbf{n}{\rm d}S_x{\rm d}t
\end{aligned}
\tag{3.7}
$$

holds for a.a. $\tau \in (0,T)$ and any continuous Lipschitz extension $\mathbf{u}_\infty \in W^{1,\infty}(\Omega;R^d)$ of $\mathbf{u}_B$ satsifying

$$\mathrm{div}\mathbf{u}_\infty \geq 0 \quad \text{in } \hat{U}_h^- \equiv \{x \in \Omega \,|\, \mathrm{dist}(x,\partial\Omega) < h\} \quad \text{for some } h > 0. \tag{3.8}$$

In (3.7), the function $H_\varepsilon(\varrho)$ is defined by

$$H_\varepsilon(\varrho) = \varrho \int_1^\varrho \frac{p_\varepsilon(s)}{s^2} \mathrm{d}s. \tag{3.9}$$

Remark 3.2. *1.* It can be shown easily that the family of test functions b in the renormalized continuity equation (3.5) can be extended to any functions $b \in C[0,\infty) \cap C^1(0,\infty)$ satisfying

$$zb' - b \in C[0,\infty), \quad |b(z)| \leq c(1+z^{5\gamma/6}), \quad |zb'(z) - b(z)| \leq c(1+z^{\gamma/2}) \tag{3.10}$$

by the Lebesgue dominated convergence theorem.

2. Satisfaction of the continuity equation (3.3) solely in the sense of distributions (i.e. with $\varphi \in C_c^\infty(Q_T)$) by a couple $(\varrho, \mathbf{u}) \in L^\infty(0,T;L^p(\Omega)) \cap C_{\mathrm{weak}}([0,T];L^p(\Omega)) \times L^2(0,T;W^{1,2}(\Omega))$, $p > 6/5$ already guarantees that $\varrho \in C([0,T];L^1(\Omega))$ (without relabeling in the t variable). It is one of the consequences of the DiPerna–Lions transport theory, cf. [5] and see e.g. Propositions 4.2 and 4.3 in Feireisl [6] for the detailed proof adapted to this situation. Consequently $b(\varrho) \in C([0,T];L^1(\Omega))$ for any b that is globally Lipschitz on $(0,\infty)$.

3. An extension $\mathbf{u}_\infty$ of $\mathbf{u}_B$ verifying (3.8) always exists due to the following lemma (see [15, Lemma 3.3]).

Lemma 3.3. *Let $\mathbf{V} \in W^{1,\infty}(\partial\Omega;R^d)$ be a Lipschitz vector field on the boundary $\partial\Omega$ of a bounded Lipschitz domain Ω. Then there exist $\overline{h} > 0$ and a vector field*

$$\mathbf{V}_\infty \in W^{1,\infty}(R^d) \cap C_c(R^d), \quad \mathrm{div}\mathbf{V}_\infty \geq 0 \quad \text{a.e. in } \hat{U}_h \tag{3.11}$$

verifying $\mathbf{V}_\infty|_{\partial\Omega} = \mathbf{V}$, where $\hat{U}_{\overline{h}} = \{x \in R^d \,|\, \mathrm{dist}(x,\partial\Omega) < \overline{h}\}$.

4 Uniform estimates with respect to ε and limit $\varepsilon \to 0$

We introduce for further convenience

$$\mathscr{H}(\varrho) = \varrho \int_1^\varrho \frac{\mathcal{p}(s)}{s^2} \mathrm{d}s, \quad \mathscr{H}_\varepsilon(\varrho) = \varrho \int_1^\varrho \frac{\mathcal{p}_\varepsilon(s)}{s^2} \mathrm{d}s. \tag{4.1}$$

In this section, we shall prove Theorem 2.4 and Theorem 2.5 whose proofs follow the same lines. The only difference dwells in an argument to show the global integrability of the pressure sequence $p_\varepsilon(\varrho_\varepsilon)$, which needs additional hypothesis (2.16) (and an additional estimate induced by this hypothesis, see (4.5)) for the global-in-time existence result.

4.1 Uniform estimates

We start by recalling uniform estimates for the couple $(\varrho_\varepsilon, \mathbf{u}_\varepsilon)$ verifying relations (3.2)–(3.7) constructed in Theorem 3.1. In view of (2.13) (in particular, seeing that $\int_\Omega \not{p}(\varrho_\varepsilon)\mathrm{div}\mathbf{u}_\infty \ \mathrm{d}x \geq 0$) the energy inequality (3.7) in combination with the conservation of mass (3.4) yields

$$\|\mathscr{H}_\varepsilon(\varrho_\varepsilon)\|_{L^\infty(0,T;L^1(\Omega))} \leq c(\text{data}), \tag{4.2}$$

$$\|\varrho_\varepsilon \mathbf{u}_\varepsilon^2\|_{L^\infty(0,T;L^1(\Omega))} \leq c(\text{data}), \tag{4.3}$$

$$\|\mathbf{u}_\varepsilon\|_{L^2(0,T;W^{1,2}(\Omega))} \leq c(\text{data}). \tag{4.4}$$

These estimates can be derived in a standard way and an interested reader may consult [3, Section 4.3.3] for all details. If $\mathbf{u}_\infty$ satisfies, in addition, the condition (2.16), then we have also

$$\underline{d}\|\not{p}_\varepsilon(\varrho_\varepsilon)\|_{L^1((0,T)\times\mathscr{O})} \leq c(\text{data}). \tag{4.5}$$

Here and hereafter, the upper bounds of the sequences (denoted usually by c) depend always tacitly on the fixed parameters of the problem (as Ω, T, p, β, μ, λ) and on some variable parameters named "data", which stand for $\mathscr{M}_0, \mathscr{E}_0, \underline{\varrho}, \overline{\varrho}, \underline{\varrho}_B, \overline{\varrho}_B$, but they are always independent of ε.

We shall use only the estimates (4.2)–(4.4) as long as possible. To get (4.5), we need additional hypotheses (2.16). Later, we shall need it in order to show integrability of the pressure sequence for the global-in-time existence result.

We deduce from (2.3–2.4), (3.9) and (4.2), in particular,

$$\mathrm{esssup}_{t\in(0,T)} \int_\Omega h_\varepsilon(\varrho_\varepsilon(t,x)) \ \mathrm{d}x \leq c(\text{data}), \tag{4.6}$$

$$\|\varrho_\varepsilon\|_{L^\infty(0,T;L^\gamma(\Omega))} \leq c(\text{data}), \tag{4.7}$$

where

$$h_\varepsilon(\varrho) = \begin{cases} (\overline{\varrho} - \varrho)^{-(\beta-1)} & \text{if } \varrho \in [0, \overline{\varrho} - \varepsilon], \\ \varepsilon^{-(\beta-1)} + (\beta-1)\varepsilon^{-\beta}(\varrho - \overline{\varrho} + \varepsilon) & \text{if } \varrho \in (\overline{\varrho} - \varepsilon, \infty). \end{cases} \tag{4.8}$$

Here, we have used also the fact that $\mathscr{H}_\varepsilon(\varrho) \sim_{\varrho\to\overline{\varrho}-} h_\varepsilon(\varrho)$. By virtue of (4.3), (4.4), and (4.7)

$$\|\varrho_\varepsilon \mathbf{u}_\varepsilon\|_{L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega))} + \|\varrho_\varepsilon \mathbf{u}_\varepsilon\|_{L^2(0,T;L^{\frac{6\gamma}{\gamma+6}}(\Omega))} \leq c(\text{data}). \tag{4.9}$$

4.2 Limit in the continuity equation and boundedness of density

We deduce from the estimates (4.4) and (4.7) that

$$\begin{aligned} \mathbf{u}_\varepsilon &\rightharpoonup \mathbf{u} \quad \text{in } L^2(0,T;W^{1,2}(\Omega)), \\ \varrho_\varepsilon &\rightharpoonup_* \varrho \quad \text{in } L^\infty(0,T;L^\gamma(\Omega)), \end{aligned} \tag{4.10}$$

for a convenient subsequence (not relabeled). We also deduce from the continuity equation (3.3), in view of the estimate (4.9), that the sequence of functions $t \mapsto \int_\Omega \varrho_\varepsilon \phi \ {\rm d}x$, $\phi \in C^1_c(\Omega)$, is equi-continuous. Therefore, by the Arzela-Ascoli theorem and separability of $L^{\gamma'}(\Omega)$, we get

$$\varrho_\varepsilon \to \varrho \quad \text{in } C_{\rm weak}(0,T; L^\gamma(\Omega)) \tag{4.11}$$

and strong convergence in $L^2(0,T;W^{-1,2}(\Omega))$ in view of compact imbedding $L^2(\Omega) \hookrightarrow\hookrightarrow W^{-1,2}(\Omega)$. Consequently, we have

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \rightharpoonup \varrho \mathbf{u} \quad \text{e.g. in } L^2(0,T; L^{\frac{6\gamma}{\gamma+6}}(\Omega)).$$

This enables us to pass to the limit in the weak formulation (3.3) so that the identity

$$\begin{aligned} &\int_\Omega \varrho(\tau,\cdot)\varphi(\tau,\cdot) \ {\rm d}x - \int_\Omega \varrho_0(\cdot)\varphi(0,\cdot) \ {\rm d}x \\ &= \int_0^\tau \int_\Omega \Big(\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi \Big) \ {\rm d}x{\rm d}t - \int_0^\tau \int_{\Gamma_{\rm in}} \varrho_B \mathbf{u}_B \cdot \mathbf{n} \varphi \ {\rm d}S_x {\rm d}t \end{aligned} \tag{4.12}$$

holds for any $\tau \in [0,T]$ and $\varphi \in C^1_c([0,T] \times (\Omega \cup \Gamma_{\rm in}))$.

To conclude this subsection, we deduce from (4.6) that for all $t \in (0,T)$ and a. a. $x \in \Omega$,

$$0 \le \varrho(t,x) < \overline{\varrho}. \tag{4.13}$$

To see this, we proceed in a few steps:

1. Coming back to (4.8) we easily verify that for any $\delta > 0$, the function h_δ is convex on $[0,\infty)$. Moreover, for all ϱ, the map $\delta \mapsto h_\delta(\varrho)$ is nonincreasing in a small right neighborghood of 0. Finally, $h_\delta \in W^{1,\infty}(0,\infty)$. Therefore, we obtain that for *almost all* $t \in (0,T)$

$$\int_\Omega h_\delta(\varrho(t)) \ {\rm d}x \le \liminf_{\varepsilon \to 0} \int_\Omega h_\delta(\varrho_\varepsilon(t)) \ {\rm d}x \le \liminf_{\varepsilon \to 0} \int_\Omega h_\varepsilon(\varrho_\varepsilon(t)) \ {\rm d}x \le c({\rm data}) \tag{4.14}$$

for any fixed sufficiently small $\delta > 0$ by virtue of (4.6), (4.11), and lower weak semicontinuity of convex functionals.

2. Now we employ item 2 of Remark 3.2 to infer first $\varrho \in C([0,T];L^1(\Omega))$ and then $h_\delta(\varrho) \in C([0,T];L^1(\Omega))$. Consequently (4.14) yields

$$\int_\Omega h_\delta(\varrho(t)) \ {\rm d}x \le c({\rm data}) \quad \text{for all } t \in [0,T] \tag{4.15}$$

uniformly in δ. In particular, this yields $\varrho \le \overline{\varrho}$.

3. Finally, letting $\delta \to 0+$ in (4.15) we get by the monotone convergence theorem

$$\int_\Omega (\overline{\varrho} - \varrho)^{-(\beta-1)} \ {\rm d}x \le c({\rm data}) \quad \text{for all } t \in [0,T].$$

The latter relation yields the requested formula (4.13).

4.3 Uniform integrability of pressure

In order to pass to the limit in the weak formulation of the momentum equation (3.6), we have to improve estimates for pressure. So far, we do not even know whether the pressure is uniformly integrable in ε. In this section we are going to prove it.

A general tool to obtain these estimates is the following Bogovskii lemma (see e.g. Galdi [14] or [7, Theorem 10.11]).

Lemma 4.1. *Let Ω be a bounded Lipschitz domain. Then there exists a linear operator*

$$\mathscr{B} : \left\{ f \in C_c^\infty(\Omega; R^3) \mid \int_\Omega f \ \mathrm{d}x = 0 \right\} \mapsto C_c^\infty(\Omega; R^3)$$

satisfying the following three properties.

1. *For all $f \in C_c^\infty(\Omega; R^3)$ satisfying $\int_\Omega f \ \mathrm{d}x = 0$*

$$\mathrm{div}\mathscr{B}[f] = f.$$

2. *Let $\overline{L}^p(\Omega) := \{f \in L^p(\Omega) \mid \int_\Omega f \ \mathrm{d}x = 0\}$. The operator $\mathscr{B}$ extends to a bounded linear operator from $\overline{L}^p(\Omega)$ to $W^{1,p}(\Omega)$ for any $1 < p < \infty$. In other words, for each $1 < p < \infty$ there is $c(p) > 0$ such that for all $f \in \overline{L}^p(\Omega)$*

$$\|\mathscr{B}[f]\|_{W^{1,p}(\Omega;R^3)} \leq c(p)\|f\|_{L^p(\Omega)}.$$

3. *If $f = \mathrm{div}\mathbf{g}$ for some $\mathbf{g} \in L^q(\Omega)$, $1 < q < \infty$ with $\mathbf{g} \cdot \mathbf{n}|_{\partial\Omega} = 0$ in the sense of normal traces, then there is $c(q) > 0$ such that*

$$\|\mathscr{B}[f]\|_{L^q(\Omega;R^3)} \leq c(q)\|\mathbf{g}\|_{L^q(\Omega,R^3)}$$

for all $\mathbf{g}$ with the above properties.

We employ this lemma to construct test functions for the momentum equation. We shall use different sets of test functions for each case of Theorem 2.4 and Theorem 2.5.

4.3.1 Bogovskii type estimates under assumptions of Theorem 2.4

We take $0 < \overline{r} < \overline{\varrho}$ and fix $T > 0$ in such a way that

$$\frac{1}{|\Omega|}\Big(\int_\Omega \varrho_0 \ \mathrm{d}x + T \int_{\Gamma_{\rm in}} \varrho_B |\mathbf{u}_B \cdot \mathbf{n}| \mathrm{d}S_x \Big) = \overline{r}. \tag{4.16}$$

We choose cut-off functions $\eta \in W_0^{1,\infty}(0,T)$ with $0 \leq \eta \leq 1$ and $\psi \in C_c^1(\Omega)$ with $0 \leq \psi \leq 1$ and

$$|\{\psi = 1\}| \geq \frac{4\overline{r}}{\overline{\varrho} + 3\overline{r}}|\Omega|,$$

and then consider the following test functions

$$\varphi = \eta(t)\mathscr{B}(\psi\varrho_\varepsilon - \psi\alpha_\varepsilon), \qquad \text{where } \alpha_\varepsilon = \frac{\int_\Omega \psi\varrho_\varepsilon \ \mathrm{d}x}{\int_\Omega \psi \ \mathrm{d}x}.$$

We test φ in the momentum equation (3.6) to obtain the following identity

$$\int_{\{\varrho_\varepsilon > \frac{\overline{r}+\overline{\varrho}}{2}\}} \eta p_\varepsilon(\varrho_\varepsilon)(\psi\varrho_\varepsilon - \psi\alpha_\varepsilon)\mathrm{d}x\mathrm{d}t + \int_{\{\varrho_\varepsilon \le \frac{\overline{r}+\overline{\varrho}}{2}\}} \eta p_\varepsilon(\varrho_\varepsilon)(\psi\varrho_\varepsilon - \psi\alpha_\varepsilon)\mathrm{d}x\mathrm{d}t = \sum_{i=1}^{5} I_i \tag{4.17}$$

where

$$\begin{aligned}
I_1 &= -\int_0^T \partial_t\eta \int_\Omega \varrho_\varepsilon\mathbf{u}_\varepsilon \cdot \mathscr{B}(\psi\varrho_\varepsilon - \psi\alpha_\varepsilon) \ \mathrm{d}x\mathrm{d}t,\\
I_2 &= \int_0^T \eta \int_\Omega \varrho_\varepsilon\mathbf{u}_\varepsilon \cdot \mathscr{B}(\mathrm{div}(\varrho_\varepsilon\mathbf{u}_\varepsilon\psi)) \ \mathrm{d}x\mathrm{d}t,\\
I_3 &= -\int_0^T \eta \int_\Omega \varrho_\varepsilon\mathbf{u}_\varepsilon \cdot \mathscr{B}\Big(\varrho_\varepsilon\mathbf{u}_\varepsilon \cdot \nabla_x\psi - \frac{\psi}{\int_\Omega \psi \ \mathrm{d}x}\int_\Omega \varrho_\varepsilon\mathbf{u}_\varepsilon \cdot \nabla_x\psi \ \mathrm{d}x\Big) \ \mathrm{d}x\mathrm{d}t,\\
I_4 &= -\int_0^T \eta \int_\Omega \varrho_\varepsilon\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x\mathscr{B}(\psi\varrho_\varepsilon - \psi\alpha_\varepsilon) \ \mathrm{d}x\mathrm{d}t,\\
I_5 &= \int_0^T \eta \int_\Omega \mathbb{S}(\nabla_x\mathbf{u}_\varepsilon) : \nabla_x\mathscr{B}(\psi\varrho_\varepsilon - \psi\alpha_\varepsilon) \ \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

By virtue of (3.4) and (4.16), the first term on the left of (4.17) is bounded from below by

$$\frac{\overline{\varrho} - \overline{r}}{4}\int_0^T \eta \int_\Omega \psi p_\varepsilon(\varrho_\varepsilon) \ \mathrm{d}x\mathrm{d}t \tag{4.18}$$

while the absolute value of the second term on the left of (4.17) is bounded from above by a number independent of ε and η (dependent on p and $\overline{r}$, $\overline{\varrho}$), and absolute values of the integrals $I_1, \ldots, I_5$ are bounded by virtue of Lemma 4.1 in view of the estimates (4.3), (4.4), and (4.7) by a number independent of ε and η (if η was chosen appropriately). Taking into account the structure of pressure (2.3) and (2.4), we conclude that for any compact set $K \subset \Omega$,

$$\|p_\varepsilon(\varrho_\varepsilon)\|_{L^1(0,T;L^1(K))} \le c(\text{data}, \overline{r}, K). \tag{4.19}$$

Before closing this section, we remark that (4.19) yields, in particular, the bound

$$\int_0^T \int_{\{\varrho_\varepsilon \le \overline{\varrho}-\varepsilon\}\cap K} (\overline{\varrho} - \varrho_\varepsilon)^{-\beta} \le c(\text{data}, K). \tag{4.20}$$

Knowing (4.19), we can extend the estimates (4.19) and (4.20) up to the boundary by the similar reasoning exposed in (4.21) and (4.22). However, the local estimates (4.19) and (4.20) are enough for our purpose.

4.3.2 Bogovskii type estimates under assumptions of Theorem 2.5

First, we shall use in the momentum equation (3.6) the following test function

$$\varphi = \eta(t)\mathscr{B}(\xi), \tag{4.21}$$

where

$$\xi = \begin{cases} 1 & \text{in } \hat{U}^-, \\ -\frac{|\hat{U}^-|}{|\mathscr{O}|} & \text{in } \mathscr{O}, \\ 0 & \text{otherwise.} \end{cases}$$

Recall that $\hat{U}^-$ is an internal neighborhood of $\partial\Omega$ defined in (3.8) and $\mathscr{O}$ is an open set defined in (2.16). Notice that $h > 0$ in the definition of $\hat{U}^-$ can be chosen so small that $\overline{\hat{U}^-} \cap \overline{\mathscr{O}} = \emptyset$. Using this test function φ, we obtain that

$$\begin{aligned}\int_0^T \eta \int_{\hat{U}^-} p_\varepsilon(\varrho_\varepsilon)\mathrm{d}x\mathrm{d}t = \frac{|\hat{U}^-|}{|\mathscr{O}|}\int_0^T \eta \int_{\mathscr{O}} p_\varepsilon(\varrho_\varepsilon)\mathrm{d}x\mathrm{d}t - \int_0^T \partial_t\eta \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathscr{B}(\xi)\ \mathrm{d}x\mathrm{d}t \\ + \int_0^T \eta \int_\Omega \Big(- \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \mathscr{B}(\xi) + \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathscr{B}(\xi)\Big)\ \mathrm{d}x\mathrm{d}t.\end{aligned}$$

The right-hand side of the above identity is bounded from above by virtue of Hölder's inequality, Lemma 4.1, and the estimates (4.3)–(4.7). Consequently,

$$\int_0^T \eta \int_{\hat{U}^-} p_\varepsilon(\varrho_\varepsilon)\mathrm{d}x\mathrm{d}t \leq c(\mathrm{data}, \underline{d}). \tag{4.22}$$

We choose cut-off functions $\eta \in W_0^{1,\infty}(0,T)$ with $0 \leq \eta \leq 1$ and $\psi \in C_c^1(\Omega)$ with $0 \leq \psi \leq 1$ and

$$\psi = 1 \text{ in } \Omega \setminus \hat{U}^-,$$

and then consider the following test functions

$$\varphi = \eta(t)\mathscr{B}(\psi\varrho_\varepsilon - \alpha_\varepsilon), \qquad \text{where } \alpha_\varepsilon = \frac{1}{|\Omega|}\int_\Omega \psi\varrho_\varepsilon\ \mathrm{d}x.$$

We test φ in the momentum equation (3.6) to obtain the following identity

$$\int_0^T \eta \int_\Omega \psi p_\varepsilon(\varrho_\varepsilon)\varrho_\varepsilon\ \mathrm{d}x\mathrm{d}t = \sum_{i=0}^{7} I_i$$

where

$$I_1 = \frac{1}{|\Omega|}\int_0^T \Big[\eta(t)\int_\Omega \psi\varrho_\varepsilon \mathrm{d}x \int_{\Omega\setminus\hat{U}^-} p_\varepsilon(\varrho_\varepsilon)\Big]\mathrm{d}t,$$

$$I_2 = \frac{1}{|\Omega|}\int_0^T \Big[\eta(t)\int_\Omega \psi\varrho_\varepsilon \mathrm{d}x \int_{\hat{U}^-} p_\varepsilon(\varrho_\varepsilon)\mathrm{d}x\Big]\mathrm{d}t,$$

$$I_3 = -\int_0^T \partial_t\eta \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathscr{B}(\psi\varrho_\varepsilon - \alpha_\varepsilon)\ \mathrm{d}x\mathrm{d}t,$$

$$I_4 = \int_0^T \eta \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathscr{B}(\mathrm{div}(\varrho_\varepsilon \mathbf{u}_\varepsilon \psi))\ \mathrm{d}x\mathrm{d}t,$$

$$I_5 = -\int_0^T \eta \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathscr{B}\Big(\varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x\psi - \frac{1}{|\Omega|}\int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \psi \mathrm{d}x\Big)\ \mathrm{d}x\mathrm{d}t,$$

$$I_6 = -\int_0^T \eta \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \mathscr{B}(\psi\varrho_\varepsilon - \alpha_\varepsilon)\ \mathrm{d}x\mathrm{d}t,$$

$$I_7 = \int_0^T \eta \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : \nabla_x \mathscr{B}(\psi\varrho_\varepsilon - \alpha_\varepsilon)\ \mathrm{d}x\mathrm{d}t.$$

By using the uniform bounds derived already in (4.3)–(4.7) and (4.22), we easily verify the boundedness of the integrals $I_2, \dots, I_7$. Seeing that

$$\int_0^T \eta \int_{\Omega\setminus\hat{U}^-} p_\varepsilon(\varrho_\varepsilon)\varrho_\varepsilon \mathrm{d}x\mathrm{d}t \le \int_0^T \eta \int_\Omega \psi p_\varepsilon(\varrho_\varepsilon)\varrho_\varepsilon\ \mathrm{d}x\mathrm{d}t,$$

we conclude

$$\int_0^T \eta \int_{\Omega\setminus\hat{U}^-} p_\varepsilon(\varrho_\varepsilon)\varrho_\varepsilon \mathrm{d}t \le c(\mathrm{data}, \underline{d}) + \frac{M_\varepsilon}{|\Omega|}\int_0^T \eta(t)\int_{\Omega\setminus\hat{U}^-} p_\varepsilon(\varrho_\varepsilon)\mathrm{d}x\mathrm{d}t, \tag{4.23}$$

where

$$M_\varepsilon := \max_{t\in[0,T]}\int_\Omega \varrho_\varepsilon(t)\ \mathrm{d}x.$$

Next, we show that it implies

$$\int_0^T \eta \int_{\Omega\setminus\hat{U}^-} \varrho_\varepsilon p_\varepsilon(\varrho_\varepsilon)\mathrm{d}x\mathrm{d}t \le c(\mathrm{data}, \underline{d}). \tag{4.24}$$

Indeed:

1. From the convergence in (4.11) we get, in particular,

$$\int_\Omega \varrho_\varepsilon\ \mathrm{d}x \to \int_\Omega \varrho\ \mathrm{d}x \quad \text{in } C[0,T].$$

In view of the estimate (4.13) we find

$$M_\varepsilon = \max_{t\in[0,T]}\int_\Omega \varrho_\varepsilon(t,x)\ \mathrm{d}x < \overline{\varrho}|\Omega|. \tag{4.25}$$

Therefore, there is $\lambda > 1$ such that

$$\limsup_{\varepsilon\to 0} \lambda\frac{M_\varepsilon}{|\Omega|} < \overline{\varrho}.$$

2. Consequently,

$$\begin{aligned}\int_{\Omega_U} p_\varepsilon(\varrho_\varepsilon)\varrho_\varepsilon \mathrm{d}x &\le c(\mathrm{data}, \underline{d}) + \int_{\Omega_U} p_\varepsilon(\varrho_\varepsilon)\frac{M_\varepsilon}{|\Omega|}\mathrm{d}x \\ &\le c + \int_{\{\varrho_\varepsilon \le \lambda \frac{M_\varepsilon}{|\Omega|}\}\cap \Omega_U} p_\varepsilon(\varrho_\varepsilon)\frac{M_\varepsilon}{|\Omega|}\ \mathrm{d}x + \int_{\{\varrho_\varepsilon > \lambda \frac{M_\varepsilon}{|\Omega|}\}\cap \Omega_U} p_\varepsilon(\varrho_\varepsilon)\frac{M_\varepsilon}{|\Omega|}\ \mathrm{d}x \\ &\le c + M_\varepsilon\, p_\varepsilon\left(\lambda\frac{M_\varepsilon}{|\Omega|}\right) + \frac{1}{\lambda}\int_{\Omega_U} p(\varrho_\varepsilon)\varrho_\varepsilon \mathrm{d}x\end{aligned}$$

where $\Omega_U = \Omega \setminus \hat{U}^-$. Inserting this estimate into (4.23) yields the desired result (4.24).

Now, we may write

$$\int_{\Omega_U} p_\varepsilon(\varrho_\varepsilon)\mathrm{d}x \le \int_{\Omega_U\cap\{\varrho_\varepsilon\le\overline{\varrho}/2\}} p_\varepsilon(\varrho_\varepsilon)\mathrm{d}x + \frac{2}{\overline{\varrho}}\int_{\Omega_U\cap\{\varrho_\varepsilon>\overline{\varrho}/2\}} p_\varepsilon(\varrho_\varepsilon)\varrho_\varepsilon\mathrm{d}x,$$

where the first integral is bounded by $p(\overline{\varrho}/2)|\Omega|$ and the integral from 0 to T of the second term on the right is uniformly bounded due to (4.24).

Using the latter analysis and (4.22), we conclude

$$\int_0^T \eta \int_\Omega p_\varepsilon(\varrho_\varepsilon)\ \mathrm{d}x\mathrm{d}t \le c(\mathrm{data}, \underline{d}). \tag{4.26}$$

Since the bound c is independent of η, we remark that (4.26) yields, in particular, the bound

$$\int_0^T \int_{\{\varrho_\varepsilon\le\overline{\varrho}-\varepsilon\}} (\overline{\varrho}-\varrho_\varepsilon)^{-\beta} \le c(\mathrm{data}). \tag{4.27}$$

4.4 Equi-integrability of pressure

Basically, the arguments for proving Theorem 2.4 and Theorem 2.5 are the same.

In order to get equi-integrability of the sequence $p_\varepsilon(\varrho_\varepsilon)$, we shall use the renormalized continuity equation (3.5). We fix the same cut-off function η as in the previous section and $0 \le \psi \in C^1_c(\Omega)$ and consider the following test function

$$\varphi = \eta(t)\mathscr{B}(\psi b(\varrho_\varepsilon) - \alpha_\varepsilon) \quad \text{ where } \alpha_\varepsilon = \frac{1}{|\Omega|}\int_\Omega \psi b(\varrho_\varepsilon)\ \mathrm{d}x$$

and

$$b(\varrho) = \begin{cases} -\ln(\overline{\varrho}/2) & \text{if } \varrho\in[0,\overline{\varrho}/2], \\ -\ln(\overline{\varrho}-\varrho) & \text{if } \varrho\in(\overline{\varrho}/2,\overline{\varrho}-\varepsilon), \\ -\ln\varepsilon & \text{if } \varrho\in[\overline{\varrho}-\varepsilon,\infty).\end{cases}$$

We note that

$$b'(\varrho) = \frac{1}{\overline{\varrho} - \varrho} 1_{(\overline{\varrho}/2, \overline{\varrho}-\varepsilon)}(\varrho),$$

where $1_E(\rho)$ denotes the characteristic function of a set E. In view of (4.6), (4.7), (4.20), and (4.27), we notice also that for any $1 \le p < \infty$ and any compact $K \subset \Omega$,

$$\begin{aligned} \|b(\varrho_\varepsilon)\|_{L^\infty(0,T;L^p(\Omega))} &\le c(\text{data}, p), \\ \|\varrho_\varepsilon b'(\varrho_\varepsilon) - b(\varrho_\varepsilon)\|_{L^\beta((0,T)\times K)} &\le c(\text{data}, K), \\ \|\varrho_\varepsilon b'(\varrho_\varepsilon) - b(\varrho_\varepsilon)\|_{L^\infty(0,T;L^{\beta-1}(\Omega))} &\le c(\text{data}). \end{aligned} \tag{4.28}$$

We test φ in the renormalized continuity equation (3.5) to obtain the following identity

$$\int_0^T \eta \int_\Omega \psi p_\varepsilon(\varrho_\varepsilon) b(\varrho_\varepsilon) \ \mathrm{d}x\mathrm{d}t = \sum_{i=1}^7 I_i,$$

where

$$\begin{aligned}
I_1 &= \frac{1}{|\Omega|} \int_0^T \eta(t) \int_\Omega \psi b(\varrho_\varepsilon) \mathrm{d}x \int_\Omega p(\varrho_\varepsilon) \ \mathrm{d}x\mathrm{d}t, \\
I_2 &= \int_0^T \partial_t \eta \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathscr{B}(\psi b(\varrho_\varepsilon) - \alpha_\varepsilon) \ \mathrm{d}x\mathrm{d}t, \\
I_3 &= \int_0^T \eta \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathscr{B}\Big(\mathrm{div}(\psi b(\varrho_\varepsilon)\mathbf{u}_\varepsilon)\Big) \ \mathrm{d}x\mathrm{d}t, \\
I_4 &= -\int_0^T \eta \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathscr{B}\Big(b(\varrho_\varepsilon)\mathbf{u}_\varepsilon \cdot \nabla_x \psi - \frac{1}{|\Omega|} \int_\Omega b(\varrho_\varepsilon)\mathbf{u}_\varepsilon \cdot \nabla_x \psi \mathrm{d}x\Big) \ \mathrm{d}x\mathrm{d}t, \\
I_5 &= -\int_0^T \eta \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathscr{B}\Big[\psi\Big(\varrho_\varepsilon b'(\varrho_\varepsilon) - b(\varrho_\varepsilon)\Big)\mathrm{div}\mathbf{u}_\varepsilon - \frac{1}{|\Omega|} \int_\Omega \psi\Big(\varrho_\varepsilon b'(\varrho_\varepsilon) - b(\varrho_\varepsilon)\Big)\mathrm{div}\mathbf{u}_\varepsilon \ \mathrm{d}x\Big] \ \mathrm{d}x\mathrm{d}t, \\
I_6 &= \int_0^T \eta \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : \nabla_x \mathscr{B}(\psi b(\varrho_\varepsilon) - \alpha_\varepsilon) \ \mathrm{d}x\mathrm{d}t, \\
I_7 &= -\int_0^T \eta \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \mathscr{B}(\psi b(\varrho_\varepsilon) - \alpha_\varepsilon) \ \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Nowadays, it is standard that the calculation effectuated above exploits integration by parts and the renormalized equation (3.5). The function b is admissible in the renormalized continuity equation as one can check by using Remark 3.2. We easily verify in view of (4.3)–(4.7), (4.20), and (4.28) that for any $\beta > 5/2$ there is $\gamma_0 > 3/2$ (sufficiently large - $\gamma_0 \to \infty$ as $\beta \to \frac{5}{2}+$) such that absolute values of $I_1, \dots, I_7$ are bounded above by some positive constants. (The most severe constraints on the values of β and γ within these calculations are imposed in estimating the term $|I_5|$.) Effectuating this process, we obtain that for any compact set $K \subset \Omega$,

$$\|p_\varepsilon(\varrho_\varepsilon) b(\varrho_\varepsilon)\|_{L^1((0,T)\times K)} \le c(\text{data}, K). \tag{4.29}$$

Consequently, the sequence $p_\varepsilon(\varrho_\varepsilon)$ is equi-integrable in $L^1((0,T)\times K)$ and

$$p_\varepsilon(\varrho_\varepsilon) \rightharpoonup \overline{p(\varrho)} \quad \text{in } L^1((0,T)\times K) \tag{4.30}$$

for any compact set $K\subset\Omega$ at least for a chosen subsequence (not relabeled).

4.5 Momentum equation

With the help of (4.3), (4.4), and (4.29) employed in the momentum equation (3.6), we verify equicontinuity of the sequence $t\mapsto \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon(t,x)\varphi(x)\ \mathrm{d}x$, $\varphi\in C_c^1(\Omega)$ in $C[0,T]$. Thereofore, we may use the Arzela-Ascoli theorem in combination with the separability of $L^{\gamma'}(\Omega)$ to show that

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \to \varrho\mathbf{u} \quad \text{in } C_{\text{weak}}(0,T;L^\gamma(\Omega)). \tag{4.31}$$

Consequently, the compact imbedding $L^2(\Omega)\hookrightarrow\hookrightarrow W^{-1,2}(\Omega)$ in combination with the weak convergence of $\mathbf{u}_n$ in $L^2(0,T;W^{1,2}(\Omega))$ implies

$$\varrho_\varepsilon \mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon \rightharpoonup \varrho\mathbf{u}\otimes\mathbf{u} \quad \text{in } L^2(0,T;L^{6\gamma/4\gamma+3}(\Omega)). \tag{4.32}$$

Thus, letting $\varepsilon\to0$ in weak formulation of (3.6), while using (4.11)–(4.32) and (4.30), we obtain that for any $\tau\in[0,T]$ and $\boldsymbol{\varphi}\in C_c^1([0,T]\times\Omega;R^3)$,

$$\begin{aligned}&\int_\Omega \varrho\mathbf{u}(\tau,\cdot)\cdot\boldsymbol{\varphi}(\tau,\cdot)\ \mathrm{d}x - \int_\Omega \varrho_0\mathbf{u}_0(\cdot)\cdot\boldsymbol{\varphi}(0,\cdot)\ \mathrm{d}x\\ &= \int_\Omega \left(\varrho\mathbf{u}\otimes\mathbf{u}:\nabla_x\boldsymbol{\varphi} + \overline{p(\varrho)}\mathrm{div}_x\boldsymbol{\varphi}\right)\ \mathrm{d}x - \int_0^\tau\int_\Omega \mathbb{S}(\nabla_x\mathbf{u}):\nabla_x\boldsymbol{\varphi}\ \mathrm{d}x\mathrm{d}t.\end{aligned} \tag{4.33}$$

The final goal to complete the proof of main theorems is to show

$$\overline{p(\varrho)} = p(\varrho), \tag{4.34}$$

which amounts, in fact, to show that the density sequence ϱ_ε converges almost everywhere in Q_T.

4.6 Effective viscous flux

We denote by $\nabla_x\Delta^{-1}$ the pseudodifferential operator of the Fourier symbol $\frac{\mathrm{i}\xi}{|\xi|^2}$ and by $\mathscr{R}$ the Riesz transform of the Fourier symbol $\frac{\xi\otimes\xi}{|\xi|^2}$. Following Lions [17] with modified in [9], we shall use the test function

$$\varphi(t,x) = \eta(t)\psi(x)\nabla_x\Delta^{-1}(\mathscr{T}_k(\varrho_\varepsilon)\psi),\ \ \eta\in C_c^1(0,T),\ \psi\in C_c^1(\Omega)$$

in the approximating momentum equation (3.6) and the test function

$$\varphi(t,x) = \eta(t)\psi(x)\nabla_x\Delta^{-1}(\overline{\mathscr{T}_k(\varrho)}\psi),\ \ \eta\in C_c^1(0,T),\ \psi\in C_c^1(\Omega)$$

in the limiting momentum equation (4.33), subtract the resulting identities, and then perform the limit $\varepsilon \to 0$. These calculations are laborious but nowadays standard. One can find details e.g. in [9, Lemma 3.2], [20], [6] or [7, Chapter 3]) to obtain the following identity

$$\begin{aligned}
&\int_0^T \int_\Omega \eta\psi^2\Big(\overline{p(\varrho)}\ \overline{\mathscr{T}_k(\varrho)} - (2\mu+\lambda)\mathrm{div}\mathbf{u}\overline{\mathscr{T}_k(\varrho)}\Big)\ \mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^T \int_\Omega \eta\psi^2\Big(\overline{p(\varrho)\mathscr{T}_k(\varrho)} - (2\mu+\lambda)\overline{\mathscr{T}_k(\varrho)\mathrm{div}\mathbf{u}}\Big)\ \mathrm{d}x\mathrm{d}t \\
&= \int_0^T \eta \int_\Omega \psi^2\mathbf{u}\cdot\Big(\overline{\mathscr{T}_k(\varrho)}\mathscr{R}\cdot(\varrho\mathbf{u}) - \varrho\mathbf{u}\cdot\mathscr{R}(\overline{\mathscr{T}_k(\varrho)})\Big)\ \mathrm{d}x\mathrm{d}t \\
&\quad - \lim_{\varepsilon\to 0}\int_0^T \eta\int_\Omega \psi^2\mathbf{u}_\varepsilon\cdot\Big(\mathscr{T}_k(\varrho_\varepsilon)\mathscr{R}\cdot(\varrho_\varepsilon\mathbf{u}_\varepsilon) - \varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\mathscr{R}(\mathscr{T}_k(\varrho_\varepsilon))\Big)\ \mathrm{d}x\mathrm{d}t,
\end{aligned} \tag{4.35}$$

where

$$\mathscr{T}_k(z) = k\mathscr{T}(z/k) \quad \text{for } k>1, \tag{4.36}$$

and $\mathscr{T}\in C^1[0,\infty)$ is concave on $[0,\infty)$ and satisfies

$$\begin{cases}\mathscr{T}(z) = z & \text{for } z\in[0,1],\\ \mathscr{T}(z)=2 & \text{for } z\in[3,\infty).\end{cases}$$

In (4.35) and in the sequel the overlined quantities $\overline{b(\varrho,\mathbf{u})}$, resp. $\overline{b(\varrho)}$ denote $L^1(Q_T)$-weak limits of sequences $b(\varrho_\varepsilon,\mathbf{u}_\varepsilon)$ resp. $b(\varrho_\varepsilon)$ (or $b_\varepsilon(\varrho_\varepsilon)$ if this is the case).

The most non-trivial moment in this process is to show that the right-hand side of this identity vanishes. The details of this calculation and reasoning can be found in [9, Lemma 3.2], [6], [20], or [7, Chapter 3]. Consequently,

$$\overline{p(\varrho)\mathscr{T}_k(\varrho)} - \overline{p(\varrho)}\ \overline{\mathscr{T}_k(\varrho)} = (2\mu+\lambda)\Big(\overline{\mathscr{T}_k(\varrho)\mathrm{div}\mathbf{u}} - \overline{\mathscr{T}_k(\varrho)}\mathrm{div}\mathbf{u}\Big). \tag{4.37}$$

Recalling that $p = \not{p} - \mathfrak{p}$ and $\not{p}$ is non decreasing, we deduce from (4.37)

$$(2\mu+\lambda)\Big(\overline{\mathscr{T}_k(\varrho)}\mathrm{div}\mathbf{u} - \overline{\mathscr{T}_k(\varrho)\mathrm{div}\mathbf{u}}\Big) \le \overline{\mathfrak{p}(\varrho)\mathscr{T}_k(\varrho)} - \overline{\mathfrak{p}(\varrho)}\,\overline{\mathscr{T}_k(\varrho)}.$$

Consequently,

$$(2\mu+\lambda)\int_0^\tau\int_\Omega\Big(\varrho\mathrm{div}\mathbf{u} - \overline{\varrho\mathrm{div}\mathbf{u}}\Big)\ \mathrm{d}x\mathrm{d}t \le \int_0^\tau\int_\Omega\Big(\overline{\mathfrak{p}(\varrho)\varrho} - \overline{\mathfrak{p}(\varrho)}\,\varrho\Big)\ \mathrm{d}x\mathrm{d}t, \tag{4.38}$$

where we have used

$$\limsup_{k\to\infty}\|\overline{\mathscr{T}_k(\varrho)} - \varrho\|_{L^1(Q_T)} = 0,$$

$$\lim_{k\to\infty}\|\overline{\mathfrak{p}(\varrho)\mathscr{T}_k(\varrho)}-\overline{\mathfrak{p}(\varrho)}\,\varrho\|_{L^1(Q_T)}=0.$$

Following Feireisl [6], we introduce the following two maps for $\varrho\in[0,\infty)$

$$\begin{aligned}&\varrho\mapsto\Lambda\varrho\ln\varrho-\varrho\mathfrak{p}(\varrho),\\&\varrho\mapsto\Lambda\varrho\ln\varrho+\mathfrak{p}(\varrho),\end{aligned}\tag{4.39}$$

which are convex provided $\Lambda>0$ is chosen large enough. Consequently,

$$\Lambda\Big(\overline{\varrho\ln\varrho}-\varrho\ln\varrho\Big)\geq\overline{\varrho\mathfrak{p}(\varrho)}-\varrho\mathfrak{p}(\varrho),\tag{4.40}$$

$$\Lambda\Big(\overline{\varrho\ln\varrho}-\varrho\ln\varrho\Big)\geq\mathfrak{p}(\varrho)-\overline{\mathfrak{p}(\varrho)}.\tag{4.41}$$

Using these inequalities and (4.38), we obtain that

$$(2\mu+\lambda)\int_0^\tau\int_\Omega\Big(\varrho\text{div}\mathbf{u}-\overline{\varrho\text{div}\mathbf{u}}\Big)\,\text{d}x\text{d}t\leq c\Lambda(1+\underline{\varrho})\int_0^\tau\int_\Omega\Big(\overline{\varrho\ln\varrho}-\varrho\ln\varrho\Big)\,\text{d}x\text{d}t.\tag{4.42}$$

4.7 Compactness of pressure

The next (and the last) step in the proof follows closely Section 4.5 in [3].

Employing Lemma 2.3 and Remark 2.2, we obtain, in particular, that $(\varrho,\mathbf{u})$ verifies

$$\begin{aligned}&\int_\Omega L(\varrho(\tau,x))\varphi(\tau,x)\text{d}x-\int_\Omega L(\varrho_0)\varphi(0,x)\text{d}x\\&=\int_0^\tau\int_\Omega\Big(L(\varrho)\mathbf{u}\cdot\nabla_x\varphi-\varphi\varrho\text{div}_x\mathbf{u}\Big)\,\text{d}x\text{d}t+\int_0^\tau\int_{\partial\Omega}L(\varrho_B)\mathbf{u}_B\cdot\mathbf{n}\varphi\text{d}S_x\text{d}t\end{aligned}\tag{4.43}$$

with any $\varphi\in C^1_c(\Omega\cup\Gamma_{\text{in}})$, where $L(\varrho)=\varrho\ln\varrho$.

Considering (3.5) with $b(\varrho_\varepsilon)=L(\varrho_\varepsilon)=\varrho_\varepsilon\ln\varrho_\varepsilon$ and letting $\varepsilon\to0$, we get

$$\begin{aligned}&\int_\Omega\overline{L(\varrho(\tau))}\varphi\text{d}x-\int_\Omega L(\varrho_0)\varphi(0,x)\text{d}x\\&=\int_0^\tau\int_\Omega\Big(\overline{L(\varrho)}\mathbf{u}\cdot\nabla_x\varphi-\varphi\overline{\varrho\text{div}_x\mathbf{u}}\Big)\,\text{d}x\text{d}t+\int_0^\tau\int_{\partial\Omega}L(\varrho_B)\mathbf{u}_B\cdot\mathbf{n}\varphi\text{d}S_x\text{d}t,\end{aligned}$$

where $\varphi\in C^1_c(\Omega\cup\Gamma_{\text{in}})$. Subtracting (4.43) from this identity, we obtain

$$\begin{aligned}&\int_\Omega\Big(\overline{L(\varrho)}-L(\varrho)\Big)(\tau)\varphi(x)\text{d}x\\&=\int_0^\tau\int_\Omega\Big(\overline{L(\varrho)}-L(\varrho)\Big)\mathbf{u}\cdot\nabla_x\varphi\text{d}x\text{d}t-\int_0^T\int_\Omega\varphi\Big(\overline{\varrho\text{div}_x\mathbf{u}}-\varrho\text{div}\mathbf{u}\Big)\,\text{d}x\text{d}t.\end{aligned}$$

Hence, we use (4.42) to get

$$\begin{aligned} &\int_\Omega \Big(\overline{\varrho\ln\varrho} - \varrho\ln\varrho\Big)(\tau,x)\varphi(x)\ \,\mathrm{d}x + \int_0^\tau\int_\Omega \Big(\varrho\ln\varrho - \overline{\varrho\ln\varrho}\Big)\mathbf{u}\cdot\nabla_x\varphi\ \,\mathrm{d}x\mathrm{d}t \\ &\le c\Lambda(1+\underline{\varrho})\int_0^\tau\int_\Omega \Big(\overline{\varrho\ln\varrho} - \varrho\ln\varrho\Big)\ \,\mathrm{d}x\mathrm{d}t \end{aligned} \tag{4.44}$$

for any $\tau \in [0,T]$ and $\varphi \in C^1_c(\Omega\cup\Gamma_{\rm in})$ with $\varphi \ge 0$.

Since $\Gamma_{\rm out}$ is in the class C^2, the function $x \mapsto d_\Omega(x) \equiv \mathrm{dist}(x,\Gamma_{\rm out})$ belongs to $C^2(\hat{\mathscr{U}}^-_{\varepsilon_0}(\Gamma_{\rm out})\cup\Gamma_{\rm out})$ for some "small" $\varepsilon_0 > 0$ where

$$\hat{\mathscr{U}}^-_{\varepsilon_0}(\Gamma_{\rm out}) \equiv \{x = \mathbf{x}_0 - z\mathbf{n}(\mathbf{x}_0)\,|\, x_0 \in \Gamma_{\rm out},\ 0 < z < \varepsilon_0\} \cap \Omega,$$

cf. Foote [13]. Moreover, if $x \in \hat{\mathscr{U}}^-_{\varepsilon}(\Gamma_{\rm out})$, $0 < \varepsilon < \varepsilon_0$, and $\mathbf{x}_0 \in \Gamma_{\rm out}$, then $x \to \mathbf{x}_0$ implies

$$\nabla_x\,\mathrm{dist}(x,\Gamma_{\rm out}) \to -\mathbf{n}(\mathbf{x}_0).$$

Therefore, if $\varepsilon_0 > 0$ is "small", then for all $x \in \hat{\mathscr{U}}^-_{\varepsilon}(\Gamma_{\rm out})$ with $0 < \varepsilon < \varepsilon_0$,

$$\mathbf{u}_\infty \cdot \nabla_x\,\mathrm{dist}(x,\Gamma_{\rm out}) < 0, \tag{4.45}$$

where $\mathbf{u}_\infty$ is defined in (2.13). Since Ω is Lipschitz, we have also that

$$\Big|\hat{U}^-_\varepsilon(\Gamma_{\rm out})\Delta\hat{\mathscr{U}}^-_\varepsilon(\Gamma_{\rm out})\Big| \to 0 \tag{4.46}$$

as $\varepsilon \to 0$, where $A\Delta B$ denotes the symmetric difference of two sets A and B, and

$$\hat{\mathscr{U}}^-_\varepsilon(\Gamma_{\rm out}) \equiv \{x \in \Omega\,|\ \mathrm{dist}(x,\Gamma_{\rm out}) < \varepsilon\}.$$

Consider the following Lipschitz test functions in Ω

$$\varphi_\varepsilon(x) = \begin{cases} 1 & \text{if } \ \mathrm{dist}(x,\Gamma_{\rm out}) > \varepsilon, \\ \frac{1}{\varepsilon}\,\mathrm{dist}(x,\Gamma_{\rm out}) & \text{if } \ \mathrm{dist}(x,\Gamma_{\rm out}) \le \varepsilon. \end{cases} \tag{4.47}$$

By the Lebesgue dominated convergence theorem and the Hardy inequality,

$$\int_0^T\int_\Omega \Big[\varrho\ln(\varrho) - \overline{\varrho\ln(\varrho)}\Big]\,(\mathbf{u}-\mathbf{u}_\infty)\cdot\nabla_x\varphi_\varepsilon\ \,\mathrm{d}x\mathrm{d}t \to 0 \qquad \text{as } \varepsilon \to 0, \tag{4.48}$$

while, in accordance with (4.45) and (4.46),

$$\liminf_{\varepsilon\to 0}\int_0^T\int_\Omega \Big[\varrho\ln(\varrho) - \overline{\varrho\ln(\varrho)}\Big]\,\mathbf{u}_\infty\cdot\nabla_x\varphi_\varepsilon\ \,\mathrm{d}x\mathrm{d}t \ge 0. \tag{4.49}$$

Using this information we conclude from (4.44) that for $\tau \in [0,T]$,

$$\int_\Omega \Big(\overline{\varrho \ln \varrho} - \varrho \ln \varrho\Big)(\tau, x)\ \mathrm{d}x \le 0. \tag{4.50}$$

On the other hand, we have

$$\overline{\varrho \ln \varrho} - \varrho \ln \varrho \ge 0 \quad \text{ a.e. in } Q_T$$

since L is convex. Thus, (4.50) yields

$$\overline{\varrho \ln \varrho} = \varrho \ln \varrho \quad \text{ a.e. in } Q_T.$$

Since the function $\varrho \mapsto \varrho \ln \varrho$ is strictly convex on $[0,\infty)$, we obtain that

$$\varrho_\varepsilon \to \varrho \quad \text{ a.e. in } Q_T \text{ and in } L^p(Q_T) \text{ for } 1 \le p < \infty, \tag{4.51}$$

cf. e.g. [7, Theorem 10.20]. We deduce from (4.51) and (4.29) that for any compact $K \subset \Omega$,

$$\mathscr{p}_\varepsilon(\varrho_\varepsilon) \to \mathscr{p}(\varrho) \text{ a.e. in } Q_T \text{ and in } L^1((0,T)\times K). \tag{4.52}$$

In particular, we have $\overline{p(\varrho)} = p(\varrho)$ in the equation (4.33).

At this stage, the most difficult part is resolved. The remaining part is to derive the energy inequality (2.8) by passing to the limit from the energy inequality (3.7).

4.8 Energy inequality

We first integrate (3.7) over $0 < \tau_1 < \tau_2 < T$ to obtain that

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2}\int_\Omega \Big(\frac{1}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon - \mathbf{u}_\infty|^2 + H(\varrho_\varepsilon)\Big)(\tau)\mathrm{d}x\mathrm{d}\tau + \int_{\tau_1}^{\tau_2}\int_0^\tau\int_\Omega \mathbb{S}(\nabla_x(\mathbf{u}_\varepsilon - \mathbf{u}_\infty)) : \nabla_x(\mathbf{u}_\varepsilon - \mathbf{u}_\infty)\mathrm{d}x\mathrm{d}t\mathrm{d}\tau \\
&\le \int_{\tau_1}^{\tau_2}\int_\Omega \Big(\frac{1}{2}\varrho_0|\mathbf{u}_0 - \mathbf{u}_\infty|^2 + H(\varrho_0)\Big)\ \mathrm{d}x\mathrm{d}\tau - \int_{\tau_1}^{\tau_2}\int_0^\tau\int_\Omega p_\varepsilon(\varrho_\varepsilon)\mathrm{div}\mathbf{u}_\infty \mathrm{d}x\mathrm{d}t\mathrm{d}\tau \\
&\quad - \int_{\tau_1}^{\tau_2}\int_0^\tau\int_{\Omega_\varepsilon} \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \mathbf{u}_\infty \cdot (\mathbf{u}_\varepsilon - \mathbf{u}_\infty)\mathrm{d}x\mathrm{d}t\mathrm{d}\tau - \int_{\tau_1}^{\tau_2}\int_0^\tau\int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_\infty) : \nabla_x(\mathbf{u}_\varepsilon - \mathbf{u}_\infty)\mathrm{d}x\mathrm{d}t\mathrm{d}\tau \\
&\quad - \int_{\tau_1}^{\tau_2}\int_0^\tau\int_{\Gamma_{\mathrm{in}}} H(\varrho_B)\mathbf{u}_B \cdot \mathbf{n}\mathrm{d}S_x\mathrm{d}t\mathrm{d}\tau - \underline{H}\int_{\tau_1}^{\tau_2}\int_0^\tau\int_{\Gamma_{\mathrm{out}}} \mathbf{u}_B \cdot \mathbf{n}\mathrm{d}S_x\mathrm{d}t\mathrm{d}\tau.
\end{aligned} \tag{4.53}$$

Now, we can use the convergences established in Section 4.2 and in (4.51) at the right-hand side and the same convergences in combination with the lower weak semi-continuity of convex functionals at the left-hand side (see e.g. [7, Theorem 10.20]). To this end, we write $H_\delta = \mathscr{H}_\delta + \mathfrak{H}$ and realize that $\mathscr{H}_\delta$ is convex and $\mathfrak{H}$ is bounded on $(0,\infty)$ so that we obtain

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2}\int_\Omega \Big(\frac{1}{2}\varrho|\mathbf{u}-\mathbf{u}_\infty|^2+H(\varrho)\Big)(\tau)\mathrm{d}x\mathrm{d}\tau+\int_{\tau_1}^{\tau_2}\int_0^\tau\int_\Omega \mathbb{S}(\nabla_x(\mathbf{u}-\mathbf{u}_\infty)):\nabla_x(\mathbf{u}-\mathbf{u}_\infty)\mathrm{d}x\mathrm{d}t\mathrm{d}\tau\\
&\le \int_{\tau_1}^{\tau_2}\int_\Omega \Big(\frac{1}{2}\varrho_0|\mathbf{u}_0-\mathbf{u}_\infty|^2+H(\varrho_0)\Big)\ \mathrm{d}x\mathrm{d}\tau-\liminf_{\varepsilon\to 0}\int_{\tau_1}^{\tau_2}\int_0^\tau\int_\Omega p_\varepsilon(\varrho_\varepsilon)\mathrm{div}\mathbf{u}_\infty\mathrm{d}x\mathrm{d}t\mathrm{d}\tau\\
&\quad-\int_{\tau_1}^{\tau_2}\int_0^\tau\int_{\Omega_\varepsilon}\varrho\mathbf{u}\cdot\nabla_x\mathbf{u}_\infty\cdot(\mathbf{u}-\mathbf{u}_\infty)\mathrm{d}x\mathrm{d}t\mathrm{d}\tau-\int_{\tau_1}^{\tau_2}\int_0^\tau\int_\Omega \mathbb{S}(\nabla_x\mathbf{u}_\infty):\nabla_x(\mathbf{u}-\mathbf{u}_\infty)\mathrm{d}x\mathrm{d}t\mathrm{d}\tau\\
&\quad-\int_{\tau_1}^{\tau_2}\int_0^\tau\int_{\Gamma_{\rm in}} H(\varrho_B)\mathbf{u}_B\cdot\mathbf{n}\mathrm{d}S_x\mathrm{d}t\mathrm{d}\tau-\underline{H}\int_{\tau_1}^{\tau_2}\int_0^\tau\int_{\Gamma_{\rm out}}\mathbf{u}_B\cdot\mathbf{n}\mathrm{d}S_x\mathrm{d}t\mathrm{d}\tau.
\end{aligned}\tag{4.54}$$

Due to (4.51) and (4.52), we have for any compact set $K\subset\Omega$

$$\int_0^\tau\int_K p^+(\varrho)\mathrm{div}\mathbf{u}_\infty\mathrm{d}x\mathrm{d}t=\lim_{\varepsilon\to0}\int_0^\tau\int_K {p_\varepsilon}^+(\varrho_\varepsilon)\mathrm{div}\mathbf{u}_\infty\mathrm{d}x\mathrm{d}t\le\liminf_{\varepsilon\to0}\int_0^\tau\int_\Omega {p_\varepsilon}^+(\varrho_\varepsilon)\mathrm{div}\mathbf{u}_\infty\mathrm{d}x\mathrm{d}t$$

while by virtue of (4.51),

$$\int_0^\tau\int_\Omega p^-(\varrho_\varepsilon)\mathrm{div}\mathbf{u}_\infty\ \mathrm{d}x\mathrm{d}t\to\int_0^\tau\int_\Omega p^-(\varrho)\mathrm{div}\mathbf{u}_\infty\ \mathrm{d}x\mathrm{d}t.$$

Inserting these observations into (4.54) yields the desired inequality (2.8).

5 On the conditions (2.13)–(2.16)

The condition (2.13) is not a necessary compatibility condition of the problem. The purpose of this section is to prove the following two claims.

1. There are domains and reasonable vector-fileds $\mathbf{u}_B$ that satisfy (2.13) and even (2.13) with (2.16).
2. If (2.13) is violated, then global-in-time weak solution to the problem (1.1)–(1.6) with (2.3)–(2.4) may not exist.

The second claim contrasts sharp with the barotropic case. In that case, the global-in-time weak solution always exists provided the addiabatic coefficient is greater than $d/2$.

5.1 Proof of the first claim

We shall show that for a sufficiently smooth simply connected domain and for the boundary velocity $\mathbf{u}_B$ with non-negative flux, there exists a suitable extension of $\mathbf{u}_B$ to Ω belonging to class (2.1) and (2.13) with (2.16). This result is formulated in the following two lemmas.

The first lemma is standard and deals with zero-flux boundary vector fields (see Finn [12], Hopf [16], or Galdi [14]).

Lemma 5.1. *Suppose that $\Omega \subset R^d$ is a bounded simply connected domain of class $C^{2+\nu}$, $\nu \in (0,1)$. Then for any $\mathbf{w} \in W^{2-\frac{1}{q},q}(\partial\Omega; R^d)$, $q > d$ satisfying*

$$\int_{\partial\Omega} \mathbf{w} \cdot \mathbf{n} \mathrm{d}S_x = 0,$$

there exists $\mathbf{W} \in W^{2,q}(\Omega)$ such that

$$\mathbf{W}|_{\partial\Omega} = \mathbf{w} \quad \textit{in the sense of traces,} \quad \mathrm{div}\mathbf{W} = 0 \tag{5.1}$$

and the map $\mathbf{w} \mapsto \mathbf{W}$ is continuous, i.e. there is $c > 0$ such that for all $\mathbf{w} \in W^{2-\frac{1}{q},q}(\partial\Omega; R^d)$,

$$\|\mathbf{W}\|_{W^{2,q}(\Omega;R^d)} \le c\|\mathbf{w}\|_{W^{2-\frac{1}{q},q}(\partial\Omega;R^d)}$$

The second lemma is less standard and deals with positive-flux boundary vector fields.

Lemma 5.2. *Suppose that Ω be a bounded simply connected domain of class $C^{2+\nu}$, $\nu \in (0,1)$, and that $\mathbf{u}_B$ is a vector field on $\partial\Omega$ in the class* (2.1) *satisfying*

$$\int_{\partial\Omega} \mathbf{u}_B \cdot \mathbf{n} \mathrm{d}S_x = K > 0. \tag{5.2}$$

Then $\mathbf{u}_B$ admits an extension to Ω satisfying (2.13) *with* (2.16).

Proof. Suppose Lemma 5.3 was established. Then there exists a vector field $\mathbf{V} \in C^2(\overline{\Omega}; R^d)$ satisfying conditions

$$\mathrm{div}\mathbf{V} \ge 0 \text{ in } \Omega, \quad \mathrm{div}\mathbf{V} > 0 \text{ in } \mathscr{O} \tag{5.3}$$

where $\mathscr{O}$ is open set such that $\overline{\mathscr{O}} \subset \Omega$ and and $\int_{\partial\Omega} \mathbf{V} \cdot \mathbf{n} \, \mathrm{d}S_x = K$. Now we can easily deduce the result of Lemma 5.2. Indeed, let $\mathbf{V}$ be the vector field constructed in Lemma 5.3 and put

$$\mathbf{w} = \mathbf{u}_B - \mathbf{V}.$$

Then we can apply Lemma 5.1 to $\mathbf{w}$ and set $\mathbf{u}_\infty = \mathbf{W} + \mathbf{V}$. This completes the proof of Lemma 5.2. □

We finish this subsection by proving the next lemma.

Lemma 5.3. *Let $\Omega \subset R^d$, $d = 2, 3$, be a bounded simply connected domain of class C^2. Given $K > 0$, there exists a vector field $\mathbf{V} \in C^2(\overline{\Omega}; R^d)$ satisfying (5.3), and*

$$\int_{\partial\Omega} \mathbf{V} \cdot \mathbf{n} \, \mathrm{d}S_x = K. \tag{5.4}$$

Proof. We first recall the following known results. Since Ω is of class C^2, there are an open neighborhood $\tilde{\mathscr{U}}$ of $\partial\Omega$ and an open connected set $C \subset \partial\Omega$, $C \subset \overline{\tilde{\mathscr{U}}^-}$, where $\tilde{\mathscr{U}}^- := \tilde{\mathscr{U}} \cap \Omega$ denotes an inner open neighborhood of $\partial\Omega$, such that the following three conditions hold.

1. For any $x \in \tilde{\mathscr{U}}^-$, there is a unique $x_B(x) \in \partial\Omega$ such that

$$d_\Omega(x) \equiv \mathrm{dist}(x; \partial\Omega) = |x - x_B(x)|,$$

and the map $x \mapsto x_B(x)$ is Lipschitz on $\tilde{\mathscr{U}}^-$.

2. The distance function d_Ω belongs to $C^2(\overline{\tilde{\mathscr{U}}^-})$.

3. $d_\Omega(x) = \mathrm{dist}(x; C)$ for all $x \in \tilde{U}_C^-$ and

$$\Delta d_\Omega(x) \leq 0 \tag{5.5}$$

whenever $x \in \tilde{\mathscr{U}}_C^- := \{x \in \tilde{\mathscr{U}}^- \,|\, x_B(x) \in C\}$.

We refer to Foote [13] for the condition 1 and 2 and to Armitage–Kuran [1] for the condition 3.

Now, we take an open connected set B (in $\partial\Omega$) with $\overline{B} \subset C$ and a function $\Lambda \in C^2(\partial\Omega)$ such that $\Lambda \in C_c^2(C)$, $\Lambda \geq 0$, $\Lambda(x) \geq k > 0$ for $x \in B$, and

$$\int_{\partial\Omega} \Lambda \; \mathrm{d}S_x = \int_{\partial\Omega} \Lambda\mathbf{n} \cdot \mathbf{n} \; \mathrm{d}S_x = K.$$

We put

$$\mathbf{V}(x) = -\Lambda(x_B(x)) h_\delta(\mathrm{dist}(x; C)) \nabla_x \, \mathrm{dist}(x; C)$$

for $x \in \Omega$ where the function h_δ is chosen in such a way that $h_\delta \in C_c^1[0, \infty)$, $h'_\delta \leq 0$, and

$$\begin{cases} h_\delta(y) = 1 - \frac{1}{\delta} y & \text{for } y \in [0, \delta/2], \\ h_\delta(y) = 0 & \text{for } y \in [\delta, \infty). \end{cases} \tag{5.6}$$

We can choose the set B and the positive number δ such that

$$\mathrm{supp}\Big(x \mapsto \Lambda(x_B(x)) h_\delta(\mathrm{dist}(x; C))\Big) \subset \tilde{\mathscr{U}}_C^- \cup C.$$

Hence $\mathrm{supp}\mathbf{V} \subset \tilde{\mathscr{U}}_C^- \cup C$. It is easy to check that $\mathbf{V} = \Lambda\mathbf{n}$ on $\partial\Omega$ and the condition (5.4) is satisfied.

Finally, using (5.5) and (5.6), we compute for $x \in \tilde{\mathscr{U}}_C^-$,

$$\begin{aligned} \mathrm{div}_x \mathbf{V}(x) &= -\Lambda(x_B(x)) h'_\delta(\mathrm{dist}(x; C)) |\nabla_x \, \mathrm{dist}(x; C)|^2 \\ &\quad - \Lambda(x_B(x)) h_\delta(\mathrm{dist}(x; C)) \Delta_x \, \mathrm{dist}(x; C) \\ &\quad - h_\delta(\mathrm{dist}(x; C)) \nabla_x (\Lambda(x_B(x))) \cdot \nabla_x \, \mathrm{dist}(x; C) \\ &\geq -\Lambda(x_B(x)) h'_\delta(\mathrm{dist}(x; C)) |\nabla_x \, \mathrm{dist}(x; C)|^2 \\ &\quad - h_\delta(\mathrm{dist}(x; C)) \nabla_x (\Lambda(x_B(x))) \cdot \nabla_x \, \mathrm{dist}(x; C) \end{aligned} \tag{5.7}$$

and for $x \in \Omega \setminus \tilde{\mathscr{U}}_C^-$,

$$\mathrm{div}_x \mathbf{V}(x) = 0.$$

Since the function $\Lambda(x_B(x))$ is constant on each segment $[x, x_B(x)]$ and $\nabla_x d_\Omega(x)$ is parallel to this segment, we obtain that for $x \in \tilde{\mathscr{U}}_C^-$,

$$\nabla_x(\Lambda(x_B(x))) \cdot \nabla_x \operatorname{dist}(x; C) = 0.$$

Moreover, we have for $x \in \tilde{\mathscr{U}}_C^-$,

$$\nabla_x \operatorname{dist}(x; C) \to -\mathbf{n}(x_B(x)).$$

Hence, if δ is sufficiently small, then there exist an open set $\mathscr{O} \subset \tilde{\mathscr{U}}_C^-$ and a positive number k such that for all $x \in \mathscr{O}$,

$$-\Lambda(x_B(x))h'_\delta(\operatorname{dist}(x; C))|\nabla_x \operatorname{dist}(x; C)|^2 > k/2 > 0$$

where we have used the continuity of all functions at the left-hand side. These facts employed in (5.7) finish the proof of Lemma 5.3. □

5.2 Proof of the second claim

We shall show that if the boundary velocity $\mathbf{u}_B$ has a negative flux over the boundary of a bounded domain, then the problem (1.1)–(1.6) with (2.3)–(2.4) may fail to have a global-in-time weak solution. The exact statement is announced in the following lemma.

Lemma 5.4. *Let $\mathbf{u}_B$ belong to the regularity class* (2.1) *and let Ω be a bounded Lipschitz domain. If*

$$\int_{\partial\Omega} \mathbf{u}_B \cdot \mathbf{n} \mathrm{d}S_x < 0,$$

then there exists ϱ_B in the class (2.1) *and $T > 0$ such that the problem* (1.1)–(1.6) *with* (2.3)–(2.4) *does not admit a global-in-time weak solution on interval* $(0, T)$.

Proof. In (2.6), we use the test function

$$\varphi = \varphi_\delta(x) = \mathfrak{h}_\delta(\operatorname{dist}(x; \Gamma_{\text{out}})), \quad x \in \Omega,$$

where $\mathfrak{h}_\delta = 1 - h_\delta$ and $h_\delta \in C_c^1[0, \infty)$, $h'_\delta \leq 0$, satisfying (5.6). Then $\mathfrak{h}_\delta \in C^1[0, \infty)$ satisfies

$$\begin{cases} \mathfrak{h}_\delta(y) = \frac{1}{\delta} y & \text{for } y \in [0, \delta/2], \\ \mathfrak{h}_\delta(y) = 1 & \text{for } y \in [\delta, \infty). \end{cases}$$

In particular, $\varphi_\delta(x) \nearrow 1$ for any $x \in \Omega$. After a straightforward manipulation, we get

$$\begin{aligned} &\int_\Omega \varrho(\tau, \cdot)\varphi_\delta \ \mathrm{d}x - \int_\Omega \varrho_0(\cdot)\varphi_\delta \ \mathrm{d}x \\ &= \int_0^\tau \int_\Omega \varrho(\mathbf{u} - \mathbf{u}_\infty) \cdot \nabla_x \varphi_\delta \ \mathrm{d}x\mathrm{d}t + \int_0^\tau \int_\Omega (\varrho - \overline{\varrho})\mathbf{u}_\infty \cdot \nabla_x \varphi_\delta \ \mathrm{d}x\mathrm{d}t \end{aligned}$$

$$+\int_0^\tau\int_\Omega \overline{\varrho}\mathbf{u}_\infty\cdot\nabla_x\varphi_\delta\ \mathrm{d}x\mathrm{d}t-\int_0^\tau\int_{\Gamma_{\rm in}}\varrho_B\mathbf{u}_B\cdot\mathbf{n}\varphi_\delta\ \mathrm{d}S_x\mathrm{d}t.$$

Using (4.13), (4.45), and (4.46), we obtain

$$\lim_{\delta\to 0}\int_0^\tau\int_\Omega(\varrho-\overline{\varrho})\mathbf{u}_\infty\cdot\nabla_x\varphi_\delta\ \mathrm{d}x\mathrm{d}t\geq 0.$$

By the similar way as in (4.48), we get

$$\lim_{\delta\to 0}\int_0^\tau\int_\Omega\varrho(\mathbf{u}-\mathbf{u}_\infty)\cdot\nabla_x\varphi_\delta\ \mathrm{d}x\mathrm{d}t=0.$$

Moreover,

$$\begin{aligned}&\int_\Omega\overline{\varrho}\mathbf{u}_\infty\cdot\nabla_x\varphi_\delta\ \mathrm{d}x-\int_{\Gamma_{\rm in}}\varrho_B\mathbf{u}_B\cdot\mathbf{n}\varphi_\delta\mathrm{d}S_x\\&=-\overline{\varrho}\int_\Omega{\rm div}\mathbf{u}_\infty\varphi_\delta\ \mathrm{d}x+\overline{\varrho}\int_{\partial\Omega}\mathbf{u}_B\cdot\mathbf{n}\varphi_\delta\ \mathrm{d}S_x\mathrm{d}t-\int_0^\tau\int_{\Gamma_{\rm in}}\varrho_B\mathbf{u}_B\cdot\mathbf{n}\varphi_\delta\ \mathrm{d}S_x\mathrm{d}t.\end{aligned}$$

Consequently, letting $\delta\to 0$ yields

$$\int_\Omega\varrho(\tau,\cdot)\ \mathrm{d}x-\int_\Omega\varrho_0(\cdot)\ \mathrm{d}x\geq\tau\Big(-\overline{\varrho}\int_{\partial\Omega}\mathbf{u}_B\cdot\mathbf{n}\mathrm{d}S_x+\int_{\Gamma_{\rm in}}(\overline{\varrho}-\varrho_B)\mathbf{u}_B\cdot\mathbf{n}\ \mathrm{d}S_x\Big).\tag{5.8}$$

If we choose ϱ_B "sufficiently close" to $\overline{\varrho}$, then we can make the quantity in the parenthesis strictly positive. Hence there is a positive number T such that $\int_\Omega\varrho(\tau,\cdot)\ \mathrm{d}x>\overline{\varrho}|\Omega|$ for all $\tau>T$. This contradicts (4.13). This completes the proof of Lemma 5.4. □

6 Piecewise smooth domains

In many practical situations in non-zero inflow/outflow regimes, the domain occupied by the fluid does not possess C^2 regularity. A typical example is a finite cylinder with inflow and outflow boundaries, which are lower and upper discs of the boundary of the cylinder. Both existence results, Theorem 2.5 and Theorem 2.4, continue to hold in this situation.

We start with the definition of a piecewise C^2 Lipschitz domain.

1. The domain Ω is bounded Lipschitz such that

$$\partial\Omega=\overline{\Gamma}_0\cup\overline{\Gamma}_{\rm in}\cup\overline{\Gamma}_{\rm out}.\tag{6.1}$$

2. There holds

$$\begin{aligned}&\Gamma_{\mathfrak{a}}=\bigcup_{i_{\mathfrak{a}}=1}^{I_{\mathfrak{a}}}\Gamma_{i_{\mathfrak{a}}}\quad\text{where }\mathfrak{a}\text{ stands for "0", "in", "out"},\\&\overline{\Gamma}_{k_{\mathfrak{a}}}\cap\overline{\Gamma}_{l_{\mathfrak{b}}}=\emptyset\quad\text{whenever }\mathfrak{a}\in\{{\rm in},{\rm out}\},\ \mathfrak{b}\in\{{\rm in},{\rm out}\},\ k_{\mathfrak{a}}\neq l_{\mathfrak{b}}.\end{aligned}\tag{6.2}$$

Here $\Gamma_{i_{\mathfrak{a}}}$ are an open connected 2-dimensional mutually disjoint manifolds of class C^2, and "in" and "out" refer to the notation (1.6), and

$$\Gamma_0 = \mathrm{int}_2\Big(\{x \in \partial\Omega \,|\, \mathbf{u}_B \cdot \mathbf{n} = 0\}\Big). \tag{6.3}$$

3. There holds

$$\gamma_{\mathfrak{a}} \equiv \partial\Gamma_{\mathfrak{a}} = \bigcup_{k_{\mathfrak{a}}=1}^{K_{\mathfrak{a}}} \gamma_{\mathfrak{a},k_{\mathfrak{a}}}, \tag{6.4}$$

where $\gamma_{\mathfrak{a},k_{\mathfrak{a}}}$ is a closed parametrized curve in R^d of class C^2 (if $d=3$) or a point (if $d=2$) such that either $\gamma_{\mathfrak{a},k_{\mathfrak{a}}} \cap \gamma_{\mathfrak{b},l_{\mathfrak{b}}} = \emptyset$, or $\gamma_{\mathfrak{a},k_{\mathfrak{a}}} = \gamma_{\mathfrak{b},l_{\mathfrak{b}}}$.

According to [4, Theorem 2.4], Theorem 3.1 for the barotropic problem (1.1)–$(2.3)_{p=p_\varepsilon}$ for approximations still holds on piecewise C^2 domains satisfying the conditions (6.1)–(6.4). The limiting process $\varepsilon \to 0$ in Section 4 requires only a bounded Lipschitz domain with C^2 open $(d-1)$-maniflods $\Gamma_{\rm in}$ and $\Gamma_{\rm out}$. Due to this facts, Theorems 2.4 and 2.5 continue to hold also for piecwise C^2 domains. More precisely, we have the following two theorems.

Theorem 6.1. *Let Ω be a Lipschitz domain satisfying the conditions* (6.1)–(6.4)*. Suppose that the pressure p, boundary data $(\varrho_B, \mathbf{u}_B)$, and initial data $(\varrho_0, \mathbf{u}_0)$ satisfy the hypotheses of Thoerem 2.4. Then there is $T > T_{\max} > 0$ such that the problem* (1.1)–(1.6) *admits at least one renormalized bounded energy weak solution $(\varrho, \mathbf{u})$ on the time interval $(0, T)$. The value of $T_{\max}$ is given by the formula (2.15).*

Theorem 6.2. *Let Ω be a Lipschitz domain satisfying the conditions* (6.1)–(6.4) *and let $T > 0$ be an arbitrary number. Suppose that the pressure p, boundary data $(\varrho_B, \mathbf{u}_B)$, and initial data $(\varrho_0, \mathbf{u}_0)$ satisfy the hypotheses of Thoerem 2.5. Then the problem* (1.1)–(1.6) *admits at least one renormalized bounded energy weak solution $(\varrho, \mathbf{u})$ on the time interval $(0, T)$.*

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