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**Abstract**

We investigate a version of one velocity Baer-Nunziato type system with dissipation describing the motion of a mixture of two compressible fluids. We define for this system weak solutions on one hand and dissipative weak solutions on the other hand, and recall the theorem about their existence on a large time interval. We investigate strong solutions and show their existence on a short time interval. Finally, we prove that any weak solution satisfies a relative energy inequality and prove for this system the weak-strong uniqueness principle. This is the main result of the paper.

**Keywords:** One velocity Baer-Nunziato type system, Weak solutions, Strong solutions, Relative energy, Weak-strong uniqueness

## 1 Introduction

There is no general agreement about the modeling of the mixture of several compressible fluids, and from the general point of view, about the two phase flow modeling. One of the acceptable models is the so called *two velocity Baer-Nunziato model.* The equations of the Baer-Nunziato model with dissipation [3], [18, Section 1] are as follows:

$$\partial_t \alpha_\pm + \mathbf{v}_I \cdot \nabla \alpha_\pm = 0, \tag{1.1}$$

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$$\partial_t(\alpha_\pm\varrho_\pm) + \mathrm{div}(\alpha_\pm\varrho_\pm\mathbf{u}_\pm) = 0, \tag{1.2}$$

$$\begin{aligned}\partial_t(\alpha_\pm\varrho_\pm\mathbf{u}_\pm) + \mathrm{div}(\alpha_\pm\varrho_\pm\mathbf{u}_\pm\otimes\mathbf{u}_\pm) + \nabla(\alpha_\pm P_\pm(\varrho_\pm)) - P_I\nabla(\alpha_\pm) \\ = \alpha_\pm\mu_\pm(\Delta\mathbf{u}_\pm) + \alpha_\pm(\mu_\pm+\lambda_\pm)\nabla\mathrm{div}\mathbf{u}_\pm\end{aligned} \tag{1.3}$$

$$0 \le \alpha_\pm \le 1, \quad \alpha_+ + \alpha_- = 1. \tag{1.4}$$

In the above $(\alpha_\pm, \alpha_\pm\varrho_\pm \ge 0, \mathbf{u}_\pm \in R^d)$ -concentrations, densities, velocities of the $\pm$ species - are unknown functions of time $t \in I = (0,T)$, $T > 0$, and $x \in \Omega \subset R^d$, $d = 2,3$, while $\mu_\pm > 0$, $\lambda_\pm + \frac{2}{d}\mu_\pm \ge 0$ are given constant shear and bulk viscosities of the $\pm$ species, $P_\pm$ are two (different) given functions defined on $[0,\infty)$ and $P_I$, $\mathbf{v}_I$ are conveniently chosen quantities - they represent the pressure and the velocity at the interface. In the multifluid modeling, there are many possibilities how the quantities $\mathbf{v}_I$, $P_I$ could be chosen, and there is no consensus about this choice. We refer the reader to [5] for the overview of multifluid models from the mathematical point of view and to [10], [19] for the physical background of the multifluid modeling.

In [22] the authors consider the Baer-Nunziato system with

$$\mu_\pm := \mu,\ \lambda_\pm := \lambda,\ \mathbf{v}_I = \mathbf{u}_\pm := \mathbf{u} \tag{1.5}$$

$$\alpha P_\pm(s) = \mathscr{P}_\pm(f_\pm(\alpha)s) \text{ for all } \alpha \in (0,1),\ s \in [0,\infty), \tag{1.6}$$

where $\mathscr{P}_\pm$ are given functions defined on $[0,\infty)$ and $f_\pm$ are given functions defined on $(0,1)$.

With this choice, the two velocity Baer-Nunziato system reduces to the following system (which we will call the *one velocity Baer-Nunziato type system*):

$$\partial_t\alpha + (\mathbf{u}\cdot\nabla)\alpha = 0, \tag{1.7}$$

$$\partial_t\varrho + \mathrm{div}(\varrho\mathbf{u}) = 0, \tag{1.8}$$

$$\partial_t z + \mathrm{div}(z\mathbf{u}) = 0, \tag{1.9}$$

$$\partial_t((\varrho+z)\mathbf{u}) + \mathrm{div}((\varrho+z)\mathbf{u}\otimes\mathbf{u}) + \nabla p(f(\alpha)\varrho, g(\alpha)z) = \mathrm{div}\mathbb{S}(\nabla\mathbf{u}), \tag{1.10}$$

in $I \times \Omega$, where $\Omega \subset R^d$, $d = 2,3$. Here $p : [0,\infty)^2 \mapsto [0,\infty)$ as well as $f, g : [0,1] \mapsto [0,\infty)$ are given functions,

$$\mathbb{S}(\nabla\mathbf{u}) = \mu(\nabla\mathbf{u} + \nabla^T\mathbf{u}) + \lambda\mathrm{div}\mathbf{u}\mathbb{I}$$

is the stress tensor with $\mathbb{I}$ the identity tensor in $R^d$. The constant viscosity coefficients satisfy the standard physical assumptions, $\mu > 0$, $\lambda + \frac{2}{d}\mu \ge 0$. The system is endowed with the initial conditions

$$\alpha|_{t=0} = \alpha_0,\ \varrho|_{t=0} = \varrho_0,\ z|_{t=0} = z_0,\ (\varrho+z)\mathbf{u}|_{t=0} = (\varrho_0+z_0)\mathbf{u}_0, \tag{1.11}$$

and the no slip boundary conditions,

$$\mathbf{u}|_{\partial\Omega} = \mathbf{0}. \tag{1.12}$$

Indeed, under the assumptions (1.5–1.6), if we set

$$\alpha = \alpha_+,\ \varrho = \alpha_+\varrho_+,\ z = \alpha_-\varrho_-,$$

in the two velocity Baer-Nunziato system (1.1–1.4), we obtain equations (1.7–1.10) with

$$p(R,Z)=\mathscr{P}_+(R)+\mathscr{P}_-(Z),\ f(s)=\frac{f_+(s)}{s},\ g(s)=\frac{f_-(1-s)}{1-s}.$$

In fact, equations (1.1), $(1.2)_\pm$ written in the new variables yield (1.7–1.9) while equation (1.10) is obtained as the sum of the momentum equations for the species $\pm$, $(1.3)_\pm$.

We notice, that assumption (1.6) is certainly true in the classical situation of two isentropic gases when

$$P_\pm(s)=a_\pm s^{\gamma^\pm},\ \gamma_\pm>0; \tag{1.13}$$

indeed, in this case

$$p(R,Z)=a_+R^{\gamma^+}+a_-Z^{\gamma^-},\ f(s):=s^{\frac{1-\gamma^+}{\gamma^+}},\ g(s)=(1-s)^{\frac{1-\gamma^-}{\gamma^-}}. \tag{1.14}$$

This gives at least one example among others when the two velocity Baer-Nunziato system under hypotheses (1.5) reduces to the one velocity Baer-Nunziato type system (1.7–1.10).

It is not without interest that equations (1.7–1.10) correspond to the barotropic and viscous version of the five-equation model of two phase flows derived by Allaire, Clerc and Kokh in [1], [2] via different considerations.

The aim of the paper is to investigate the stability issues for a version of the one velocity Baer-Nunziato system (1.7–1.12). We prove that strong solutions (if they exist on the given time interval) are stable in the class of weak solutions, provided the constitutive law for the pressure gives rise to a strictly convex Helmohltz function. In particular, under such circumstances, the weak-strong uniqueness principle holds: any weak solution emanating from the sufficiently regular data coincides with the strong solution emanating from the same initial data as long as the latter exists.

The organization of the paper is inspired by [13], where the weak strong uniqueness is investigated in the "simple" mono-fluid case: In Section 2 we introduce the weak solutions, the relative energy functional and the dissipative solutions for the system (1.7–1.12). In Section 3 we announce the theorem about global existence of weak solutions, cf. Theorem 3.1, in the form proved in [22]. In Section 4 we show the existence of strong solutions for the system (1.7–1.12) at least on a short time interval, see Theorem 4.1. Any weak solution with square integrable densities satisfies the relative energy inequality with arbitrary test functions and is therefore dissipative. This statement is announced in Theorem 5.2 and proved in Section 5. If we take in the relative energy inequality as a test function the strong solution of the same system, the inequality takes a particular form with the remainder which is quadratic in the difference of weak and strong solutions. This statement is formulated in Theorem 6.1 and proved in Section 6. The inequality derived in Theorem 6.1 is employed in Section 7 to show the weak strong uniqueness principle for the system (1.7–1.12), see Theorem 7.2.

We close the introduction by introducing the basic notation and the functional spaces used throughout this paper. In what follows, the scalar-valued functions will be printed with the usual font, the vector-valued functions will be printed in bold, and the tensor-valued functions with a special font, i.e. $\varrho$ stands for the density, $\mathbf{u}$ for the velocity field and $\mathbb{S}$ for the stress tensor. We use standard notation for the Lebesgue and Sobolev spaces equipped by the standard norms $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{W^{k,p}(\Omega)}$, respectively. We will sometimes distinguish the scalar-, the vector- and the tensor-valued functions in the notation, i.e. we use $L^p(\Omega)$ for scalar quantities, $L^p(\Omega; R^3)$ for vectors and $L^p(\Omega; R^{3\times 3})$ for tensors. The indication of the $R$ or tensor character of the fields (here $; R^3$ or $; R^{3\times 3}$) may be omitted, when there is no danger of confusion. The Bochner spaces of integrable functions on $I$ with values in a Banach space $X$ will be denoted $L^p(I; X)$; likewise the spaces of continuous functions on $\overline{I}$ with values in $X$ will be denoted $C(\overline{I}; X)$. The norms in the Bochner spaces will be denoted $\|\cdot\|_{L^p(I;X)}$ and $\|\cdot\|_{C(\overline{I};X)}$, respectively. In most cases, the Banach space $X$ will be either the Lebesgue or the Sobolev space. Finally, we use vector spaces $C_{\rm weak}(\overline{I}; X)$ of functions defined on [0,T] belonging to the space $L^\infty(I; X)$ and continuous in $\overline{I}$ with respect to the weak topology of $X$ (meaning that $f \in C_{\rm weak}(\overline{I}; X)$ if $f \in L^\infty(I; X)$ and $t \mapsto \mathscr{F}(f(t))$ belongs to $C(\overline{I})$ for any $\mathscr{F} \in X^*$ ).

The generic numbers in estimates will be denoted by $c$, $\underline{c}$, $\overline{c}$, $C$, $\underline{C}$, $\overline{C}$ and their value may change even in the same formula or in the same line. They may depend on the parameters characterizing the problem; this dependence is always indicated in the text, where they appear.

# 2 Weak and dissipative solutions

We begin with the definition of weak solutions to system (1.7–1.12).

**Definition 2.1**

*Let $\mathscr{O}$ be an open subset of $(0,\infty)^2$ and let $0 \le \underline{\alpha} < \overline{\alpha} \le 1$. Suppose that $\alpha_0(x) \in [\underline{\alpha}, \overline{\alpha}]$ and $(f(\alpha_0)\varrho_0(x), g(\alpha_0)z_0(x)) \in \overline{\mathscr{O}}$ for a.a. $x \in \Omega$. We say that the quartet $(\alpha, \varrho, z, \mathbf{u})$ is a bounded energy weak solution of problem (1.7–1.12) with densities ranging in $\overline{\mathscr{O}}$ and concentration in $[\underline{\alpha}, \overline{\alpha}]$ if:*

1. *It belongs to functional spaces:*

$$\alpha \in C(\overline{I}; L^1(\Omega)),\ (\varrho, z) \in C_{\rm weak}([0,T]; L^\gamma(\Omega)) \cap C(\overline{I}, L^1(\Omega)),\ \gamma > 1,\ \mathbf{u} \in L^2(0,T; W_0^{1,2}(\Omega; R^d)),$$
$$\text{for all } t \in \overline{I},\ \underline{\alpha} \le \alpha(t,x) \le \overline{\alpha},\ (f(\alpha)\varrho(t,x), g(\alpha)z(t,x)) \in \overline{\mathscr{O}} \ \text{ for a.a. } x \in \Omega. \tag{2.1}$$

2. *The integral identity*

$$\int_\Omega r(\tau,\cdot)\varphi(\tau,\cdot)\ {\rm d}x - \int_\Omega r_0(\cdot)\varphi(0,\cdot)\ {\rm d}x = \int_0^\tau \int_\Omega \Big( r\partial_t\varphi + r\mathbf{u}\cdot\nabla\varphi \Big)\ {\rm d}x{\rm d}t \tag{2.2}$$

*holds for any $\tau \in [0,T]$ and $\varphi \in C_c^1([0,T] \times \overline{\Omega})$, where $r$ stands for $\varrho$ and $z$.*

4. *The integral identity*

$$\int_\Omega \alpha(\tau,\cdot)\varphi(\tau,\cdot)\ {\rm d}x - \int_\Omega \alpha_0(\cdot)\varphi(0,\cdot)\ {\rm d}x = \int_0^\tau \int_\Omega \Big( \alpha\partial_t\varphi + \alpha\mathbf{u}\cdot\nabla\varphi + \alpha\varphi{\rm div}\mathbf{u} \Big)\ {\rm d}x{\rm d}t \tag{2.3}$$

*holds for any $\tau \in [0,T]$ and $\varphi \in C_c^1([0,T] \times \overline{\Omega})$.*

5. *The function* $(\varrho+z)\mathbf{u} \in C_{\rm weak}([0,T], L^{\frac{2\gamma}{\gamma+1}}(\Omega))$*, and the integral identity*

$$\int_\Omega (\varrho+z)\mathbf{u}(\tau,\cdot)\cdot\boldsymbol{\varphi}(\tau,\cdot)\ {\rm d}x - \int_\Omega (\varrho_0+z_0)\mathbf{u}_0(\cdot)\boldsymbol{\varphi}(0,\cdot)\ {\rm d}x \tag{2.4}$$

$$= \int_0^\tau \int_\Omega \Big( (\varrho+z)\mathbf{u}\cdot\partial_t\varphi + (\varrho+z)\mathbf{u}\otimes\mathbf{u} : \nabla\boldsymbol{\varphi} + p(f(\alpha)\varrho, g(\alpha)z){\rm div}_x\boldsymbol{\varphi} - \mathbb{S}(\nabla\mathbf{u}) : \nabla\boldsymbol{\varphi}\Big)\ {\rm d}x{\rm d}t$$

*holds for any* $\tau\in[0,T]$ *and any* $\boldsymbol{\varphi}\in C^1_c([0,T]\times\Omega; R^d)$.

6. *There exists a function* $H \in C(\overline{\mathscr{O}})\cap C^1(\mathscr{O})$*, a solution of first order partial differential equation*

$$R\partial_R H(R,Z) + Z\partial_Z H(R,Z) - H(R,Z) = p(R,Z) \ \textit{in}\ \mathscr{O}, \tag{2.5}$$

*such that the energy inequality*

$$\int_\Omega \Big(\frac{1}{2}(\varrho+z)|\mathbf{u}|^2 + H(f(\alpha)\varrho, g(\alpha)z)\Big)(\tau)\ {\rm d}x + \int_0^\tau\int_\Omega \mathbb{S}(\nabla\mathbf{u}) : \nabla\mathbf{u}\ {\rm d}x{\rm d}t \tag{2.6}$$

$$\le \int_\Omega \Big(\frac{1}{2}(\varrho_0+z_0)|\mathbf{u}_0|^2 + H(f(\alpha_0)\varrho_0, g(\alpha_0)z_0)\Big)\ {\rm d}x$$

*holds for a.a.* $\tau \in I$*. Function* $H$ *is called the Helmholtz function corresponding to the pressure* $p$*.*

We associate to the Helmholtz function (2.5) the relative energy function

$$E_{\alpha,\beta}(\varrho, z|\mathfrak{r},\mathfrak{z}) = E(f(\alpha)\varrho, g(\alpha)z|f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z}) \tag{2.7}$$

where

$$E(R,Z|\mathfrak{R},\mathfrak{Z}) = H(R,Z) - \partial_R H(\mathfrak{R},\mathfrak{Z})(R-\mathfrak{R}) - \partial_Z H(\mathfrak{R},\mathfrak{Z})(Z-\mathfrak{Z}) - H(\mathfrak{R},\mathfrak{Z}).$$

**Definition 2.2**
*We say that the quartet* $(\alpha,\varrho,z,\mathbf{u})$ *is a dissipative weak solution to the one velocity Baer-Nunziato type system (1.7 - 1.12) with densities in* $\overline{\mathscr{O}}$ *and concentration in* $[\underline{\alpha},\overline{\alpha}]$ *if :*

1. *It fulfills all statements in Items 1.–5. in the Definition 2.1;*

2. *It satisfies the relative energy inequality,*

$$\int_\Omega \left(\frac{1}{2}(\varrho+z)|\mathbf{u}-\mathbf{U}|^2 + E_{\alpha(\tau),\beta(\tau)}(\varrho,z|\mathfrak{r},\mathfrak{z})\right)(\tau,\cdot)\ {\rm d}x \tag{2.8}$$

$$+\int_0^\tau\int_\Omega \mathbb{S}(\nabla\mathbf{u}-\nabla\mathbf{U}) : (\nabla\mathbf{u}-\nabla\mathbf{U})\ {\rm d}x{\rm d}t$$

$$\le \int_\Omega \left(\frac{1}{2}(\varrho_0+z_0)|\mathbf{u}_0-\mathbf{U}(0,\cdot)|^2 + E_{\alpha_0,\beta(0)}(\varrho_0,z_0|\mathfrak{r}(0,\cdot),\mathfrak{z}(0,\cdot))\right)\ {\rm d}x$$

$$+\int_0^\tau \mathscr{R}_{\alpha,\beta}(\varrho, z, \mathbf{u}|\mathfrak{r}, \mathfrak{z}, \mathbf{U})dt$$

*for a.a.* $\tau \in (0,T)$ *with any*

$$(\beta, \mathfrak{r}, \mathfrak{z}, \mathbf{U}) \in C^1_c([0,T]\times\overline{\Omega}; R^6),\ \underline{\alpha} \le \beta \le \overline{\alpha},\ (\mathfrak{r}(t,x), \mathfrak{z}(t,x)) \in \overline{\mathscr{O}},\ \mathbf{U}|_{\partial\Omega} = 0, \tag{2.9}$$

*where*

$$\mathscr{R}_{\alpha,\beta}(\varrho, z, \mathbf{u}|\mathfrak{r}, \mathfrak{z}, \mathbf{U})$$
$$= \int_\Omega \Big[(\mathbf{U}-\mathbf{u})\cdot\Big((\varrho+z)\Big(\partial_t\mathbf{U} + (\mathbf{u}\cdot\nabla)\mathbf{U}\Big)\Big) + \mathbb{S}(\nabla\mathbf{U}) : \nabla(\mathbf{U}-\mathbf{u})\Big]dx$$
$$+\int_\Omega \Big(p(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z}) - p(f(\alpha)\varrho, g(\alpha)z)\Big)\mathrm{div}\mathbf{U}dx$$
$$+\int_\Omega \Big(f(\beta)\mathfrak{r} - f(\alpha)\varrho\Big)\partial_t\partial_R H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})dx$$
$$+\int_\Omega \Big(g(\beta)\mathfrak{z} - g(\alpha)z\Big)\partial_t\partial_Z H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})dx$$
$$+\int_\Omega \Big(f(\beta)\mathfrak{r}\mathbf{U} - f(\alpha)\varrho\mathbf{u}\Big)\cdot\nabla\partial_R H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})dx$$
$$+\int_\Omega \Big(g(\beta)\mathfrak{z}\mathbf{U} - g(\alpha)z\mathbf{u}\Big)\cdot\nabla\partial_Z H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})dx.$$

# 3 Existence of weak solutions

Starting from now, we shall limit ourselves to the three dimensional flows (meaning that $d=3$). The treatment of the situation with $d=2$ is similar; we let its details to the interested reader.

Let $0<\underline{\alpha}<\overline{\alpha}<1$, $0<\underline{a}<\overline{a}$, and set

$$\mathscr{O} = \mathscr{O}_{\underline{a},\overline{a}} := \{(R,Z)\in R^2 \,|\, R>0,\ \underline{a}R < Z < \overline{a}R\} \tag{3.1}$$

The existence of bounded energy weak solutions to problem (1.7–1.12) with the densities in the range $\overline{\mathscr{O}}_{\underline{a},\overline{a}}$ and the concentration in the range $[\underline{\alpha},\overline{\alpha}]$ has been proved in [22, Theorem 1], in particular, under the following hypotheses:

1. *Hypothesis on the domain*

$$\Omega \subset R^3 \text{ is a bounded domain in the regularity class } C^{2,\nu},\ \nu\in(0,1). \tag{3.2}$$

2. *Hypotheses on initial data:*

$$(f(\alpha_0)\varrho_0(x), g(\alpha_0)z_0(x)) \in \overline{\mathscr{O}}_{\underline{a},\overline{a}},\ \alpha_0(x)\in[\underline{\alpha},\overline{\alpha}], \tag{3.3}$$

$$\varrho_0 \in L^\gamma(\Omega),\ z_0\in L^\beta(\Omega)\ \text{ if } \beta>\gamma,\ (\varrho_0+z_0)|\mathbf{u}_0|^2 \in L^1(\Omega). \tag{3.4}$$

3. *Regularity and growth of the pressure function:* Pressure function is such that

$$p \in C([0,\infty)^2) \cap C^1((0,\infty)^2), \tag{3.5}$$

and there is a number $C \geq 1$ (dependent on $\underline{\alpha}$, $\overline{\alpha}$, $\underline{a}$ $\overline{a}$) such that for all $(R,Z) \in \mathscr{O}_{\underline{a},\overline{a}}$

$$C^{-1}(R^\gamma + Z^\beta - 1) \leq p(R,Z) \leq C(R^\gamma + Z^\beta + 1), \tag{3.6}$$

with $\gamma \geq \frac{9}{5}$, $\beta > 0$. Moreover

$$|\partial_Z p(R,Z)| \leq C(R^{-\underline{\Gamma}} + R^{\overline{\Gamma}-1}) \text{ in } \mathscr{O}_{\underline{a},\overline{a}} \tag{3.7}$$

with some $0 \leq \underline{\Gamma} < 1$, and with some $0 < \overline{\Gamma} < \gamma + \gamma_{BOG}$, where $\gamma_{\rm BOG} = \min\{\frac{\gamma}{2}, \frac{2}{3}\gamma - 1\}$. Finally, the functions $\varrho \mapsto p(\varrho, Z)$, $Z > 0$ resp. $Z \mapsto \partial_Z p(\varrho, Z)$, $\varrho > 0$ are Lipschitz on $(Z/\overline{a}, Z/\underline{a}) \cap (\underline{r}, \infty)$ resp. $(\underline{a}\varrho, \overline{a}\varrho) \cap (\underline{r}, \infty)$ for all $\underline{r} > 0$ with the Lipschitz constants

$$\widetilde{L}_p \leq C(\underline{r})(1 + Z^A) \text{ resp. } \widetilde{L}_P \leq C(\underline{r})(1 + \varrho^A) \tag{3.8}$$

with some non negative number $A$. The number $C(\underline{r})$ may diverge to $+\infty$ as $\underline{r} \to 0^+$.

4. *Structure of the pressure:* It is assumed that

$$p(R, Rs) = \mathscr{P}(R,s) - \mathscr{R}(R,s), \tag{3.9}$$

where $[0,\infty) \ni R \mapsto \mathscr{P}(R,s)$ is non decreasing for any $s \in [\underline{a}, \overline{a}]$, and $R \mapsto \mathscr{R}(R,s)$ is for any $s \in [\underline{a}, \overline{a}]$ a non-negative $C^2$-function in $[0,\infty)$ with $C^2$-norm uniformly bounded with respect to $s \in [\underline{a}, \overline{a}]$, with a compact support uniform with respect to $s \in [\underline{a}, \overline{a}]$. Moreover, if $\gamma = \frac{9}{5}$,

$$\mathscr{P}(R,s) = \pi(s) R^\gamma + \mathscr{p}(R,s), \tag{3.10}$$

where $[0,\infty) \ni R \mapsto \mathscr{p}(R,s)$ is non decreasing for any $s \in [\underline{a}, \overline{a}]$ and $\pi \in L^\infty(\underline{a}, \overline{a})$, $\operatorname{ess\,inf}_{s\in(\underline{a},\overline{a})} \pi(s) \geq \underline{\pi} > 0$. Finally,

$$\forall R \in (0,1), \ \sup_{s\in[0,\overline{a}]} P(R, Rs) \leq cR^B \text{ with some } c > 0 \text{ and } B > 0. \tag{3.11}$$

5. *Regularity and monotonicity of functions $f$ and $g$:* Functions $f, g \in C^1((0,1))$ and they are both strictly monotone and non vanishing on the interval $(0,1)$.

The theorem on existence of weak solutions to the one velocity Baer-Nunziato type system (1.7–1.12) reads (cf. [22, Theorem 1]):

**Theorem 3.1.** *Under assumptions enumerated in items 1.-5. above, problem (1.7–1.12) admits at least one bounded energy weak solution $(\alpha, \varrho, z, \mathbf{u})$ with the densities ranging in $\overline{\mathscr{O}}_{\underline{a},\overline{a}}$ and the concentration in $[\underline{\alpha}, \overline{\alpha}]$ in the sense of Definition 2.1, where $\mathscr{O}_{\underline{a},\overline{a}}$ is defined in (3.1), satisfying moreover: for all $t \in \overline{I}$, $(f(\alpha)\varrho(t,x), g(\alpha)z(t,x)) \in \overline{\mathscr{O}}$ for a.a. $x \in \Omega$, for all $t \in \overline{I}$, $\underline{\alpha} \leq \alpha(t,x) \leq \overline{\alpha}$ for a.a. $x \in \Omega$, $\alpha, \varrho, z \in C([0,T]; L^1(\Omega))$, $\varrho, z \in L^2(Q_T)$, $z \in C_{weak}([0,T); L^\beta(\Omega))$ if $\beta > \gamma$, and $P(\varrho, Z) \in L^q(I \times \Omega)$ for some $q > 1$.*

**Remark 3.1**

1. A convenient Helmholtz function $H(R,Z)$ corresponding to $p$ in the energy inequality (2.6) in Theorem 3.1 can be calculated from the explicit formula

$$H(R,Z) = R\int_1^R \frac{p(s, s\frac{Z}{R})}{s^2}\mathrm{d}s, \text{ if } R\neq 0,\ H(0,0)=0. \tag{3.12}$$

   We notice that condition (3.11) guarantees namely the continuity of $H$ in $\overline{\mathcal{O}}_{\underline{a},\overline{a}}$ (i.e. notably at $(0,0)$) and assumption (3.6) guarantees that its growth is the same as that one of $p$:

$$\underline{C}(R^\gamma + Z^\beta - 1) \leq H(R,Z) \leq \overline{C}(R^\gamma + Z^\beta + 1) \text{ in } \mathcal{O}_{\underline{a},\overline{a}}. \tag{3.13}$$

2. Pressure function $p$ introduced in (1.14) and corresponding functions $f, g$ originated in $P_\pm$ introduced in formula (1.13) represent one example (among others) which satisfies all assumptions on $p, f, g$ requested by Theorem 3.1.

3. The domain $\Omega$ in Theorem 3.1 can be taken Lipschitz. See [16] for the methods allowing this generalization.

4. Theorem 3.1 holds also with slip (Navier) boundary conditions

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0,\ (\mathbb{S}(\nabla\mathbf{u})\mathbf{n})\times\mathbf{n}|_{\partial\Omega} = 0$$

   instead of (1.12), provided the definition of weak solutions is modified accordingly, cf. [22]. It also holds, with appropriate modifications, provided $\Omega$ is a periodic cell.

5. Existence theorems for (even simple) multifluid models are in a short supply in the mathematical literature. Theorem 3.1 is one of such examples. We refer to [21], [25], [6], [24] for another relevant examples.

# 4 Local in time existence of a strong solution

The final goal of the paper is to compare the weak solutions of the system (1.7–1.12) on (an arbitrary large) time interval $[0,T)$ - whose existence, under certain assumptions, is guaranteed by Theorem 3.1 - with a strong solution of the same system on the same time interval, provided the latter exists. The existence of strong solutions for the system (1.7–1.12) on an arbitrary large time interval is, however, not known. The aim of this section is to show that the strong solutions exist at least on a short time interval. This result is the subject of the following theorem.

**Theorem 4.1.** *Let $\Omega\in C^3$ be a bounded domain, $0<\underline{\mathfrak{r}}<\overline{\mathfrak{r}}<\infty$, $0<\underline{\mathfrak{z}}<\overline{\mathfrak{z}}<\infty$, $0<\underline{\beta}<\overline{\beta}<1$ be constants. Assume that*

$$p\in C^2((0,\infty)^2),\ f,g\in C^2((0,1)) \textit{ are non vanishing on } (0,1).$$

*Suppose that*

$$\mathbf{u}_0 \in W_0^{1,2}(\Omega),\ \beta_0, \mathfrak{r}_0, \mathfrak{z}_0 \in W^{2,2}(\Omega),$$
$$\underline{\beta} \le \beta_0 \le \overline{\beta},\ \underline{\mathfrak{r}} \le f(\beta_0)\mathfrak{r}_0 \le \overline{\mathfrak{r}},\ \ \underline{\mathfrak{z}} \le g(\beta_0)\mathfrak{z}_0 \le \overline{\mathfrak{z}},$$
$$\frac{1}{\mathfrak{r}_0 + \mathfrak{z}_0}\Big( - \nabla p(f(\beta_0)\mathfrak{r}_0, g(\beta_0)\mathfrak{z}_0) + \mu\Delta\mathbf{u}_0 + (\mu+\lambda)\nabla\mathrm{div}\mathbf{u}_0 - (\mathfrak{r}_0+\mathfrak{z}_0)\mathbf{u}_0\nabla\mathbf{u}_0\Big) \in W_0^{1,2}(\Omega).$$

1. *Then there exists an interval* $I_* = [0, T_*)$ *and numbers* $\underline{r}$, $\overline{r}$, $\underline{z}$, $\overline{z}$, $0 < \underline{r} < \underline{\mathfrak{r}} < \overline{\mathfrak{r}} < \overline{r} < \infty$, $0 < \underline{z} < \underline{\mathfrak{z}} < \overline{\mathfrak{z}} < \overline{z} < \infty$ *such that the problem (1.7–1.12) admits in the class*

$$(\beta, r, z) \in C(I_*; W^{2,2}(\Omega)),\ \partial_t(\beta, r, z) \in C(I_*; W^{1,2}(\Omega)), \tag{4.1}$$
$$\mathbf{u} \in L^2(I_*; W^{3,2}(\Omega; R^3)),\ \partial_t\mathbf{u} \in L^2(I_*; W^{2,2}(\Omega, R^3)),\ \partial_t^2\mathbf{u} \in L^2(I_*; L^2(\Omega, R^3)),$$
$$\underline{\beta} \le \beta \le \overline{\beta},\ \underline{r} \le f(\beta)r \le \overline{r},\ \underline{z} \le g(\beta)z \le \overline{z}, \tag{4.2}$$
$$\mathbf{u}(0) = \mathbf{u}_0,\ \mathbf{u}|_{(0,T)\times\partial\Omega} = 0$$

*a unique strong solution* $(\beta, \mathfrak{r}, \mathfrak{z}, \mathbf{u})$.

2. *If moreover*
$$\underline{b}f(\beta_0)\mathfrak{r}_0 \le g(\beta_0)\mathfrak{z}_0 \le \overline{b}g(\beta_0)\mathfrak{r}_0,$$
*with some* $0 < \underline{b} < \overline{b} < \infty$, *then*
$$\underline{b}f(\beta)\mathfrak{r} \le g(\beta)\mathfrak{z} \le \overline{b}g(\beta)\mathfrak{r}. \tag{4.3}$$

In view of what is known from the "mono-fluid" case, cf. [27], [7], Theorem 4.1 is not surprising. However, to the best of our knowledge, there is no reference to its proof, the closest relevant reference beeing [26]. We will perform the proof for the sake of completeness. In this proof, we follow closely the line of the proofs of local existence of weak solutions in the "mono-fluid" case, see e.g. [27].

**Proof of Theorem 4.1**
We shall proceed in several steps.

**Step 1:** *Strategy of the proof*
Given an interval $I = (a, b)$, $-\infty < a < b < \infty$, we define Banach spaces

$$V_{(a,b)} := \{(\beta, r, z) \in C([a, b]; W^{2,2}(\Omega)),\ \partial_t(\beta, r, z) \in C([a, b]; W^{1,2}(\Omega))\}$$
$$Y_{(a,b)} := \{\mathbf{u} \in L^2(a, b; W^{3,2}(\Omega; R^3)),\ \partial_t\mathbf{u} \in L^2(a, b; W^{2,2}(\Omega, R^3)),\ \partial_t^2\mathbf{u} \in L^2(a, b; L^2(\Omega, R^3))\}$$

(they are endowed with the natural norms denoted $\|\beta, \varrho, z\|_{V_{(a,b)}}$ and $\|\mathbf{u}\|_{Y_{(a,b)}}$, respectively) and their (closed, convex) subsets

$$\mathscr{V}_{(a,b)} := \{(\beta, r, z) \in V_{(a,b)} \,|\, \underline{\beta} \le \beta \le \overline{\beta},\ \underline{r} \le f(\beta)r \le \overline{r},\ \underline{z} \le g(\beta)z \le \overline{z}\},$$
$$\mathscr{Y}_{(a,b),B} = \{\mathbf{u} \in Y_{(a,b)} \,|\, \mathbf{u}(0) = \mathbf{u}_0,\ \mathbf{u}|_{(a,b)\times\partial\Omega} = 0, \|\mathbf{u}\|_{Y_{(a,b)}} \le B\}.$$

In the above, the number $B > 0$ (and $T > 0$) will be specified later.

1. Let $B > 0$ be fixed and suppose that $\tilde{\mathbf{u}} \in \mathscr{Y}_{(0,T),B}$, $T > 0$ is given. We shall first solve the following system of transport equations:

$$\partial_t \tilde{\beta} + \tilde{\mathbf{u}} \cdot \nabla \tilde{\beta} = 0, \tag{4.4}$$

$$\partial_t \tilde{\varrho} + \operatorname{div}(\tilde{\varrho} \tilde{\mathbf{u}}) = 0, \tag{4.5}$$

$$\partial_t \tilde{z} + \operatorname{div}(\tilde{z} \tilde{\mathbf{u}}) = 0 \tag{4.6}$$

   with the initial conditions

$$\tilde{\beta}|_{t=0} = \beta_0, \ \ \tilde{\varrho}|_{t=0} = \mathfrak{r}_0, \ \ \tilde{z}|_{t=0} = \mathfrak{z}_0, \tag{4.7}$$

   for the unknown functions $(\tilde{\beta}, \tilde{\varrho}, \tilde{z}) \in V_{(0,T)}$. We shall show that there is a unique solution $(\tilde{\beta}, \tilde{\varrho}, \tilde{z}) \in \mathscr{V}_{(0,T)}$ of equations (4.4–4.6) emanating from the initial conditions $(\beta_0, \mathfrak{r}_0, \mathfrak{z}_0)$.

2. We shall next find $\mathbf{u} \in Y_{(0,T)}$ a solution of the following parabolic system:

$$(\tilde{\varrho} + \tilde{z}) \partial_t \mathbf{u} - \mu \Delta \mathbf{u} + (\mu + \lambda) \nabla \operatorname{div} \mathbf{u} = \mathbf{F} \tag{4.8}$$

   with the right hand side given by

$$\mathbf{F} = -(\tilde{\varrho} + \tilde{z}) \tilde{\mathbf{u}} \cdot \nabla \tilde{\mathbf{u}} - \nabla p(f(\tilde{\alpha}) \tilde{\varrho}, g(\tilde{\alpha}) \tilde{z}), \tag{4.9}$$

   with the initial conditions

$$\mathbf{u}|_{t=0} = \mathbf{u}_0, \tag{4.10}$$

   and the boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0. \tag{4.11}$$

3. Next, we shall consider the map

$$\Phi : \mathscr{Y}_{(0,T),B} \to C([0,T]; L^2(\Omega)), \ \mathbf{u} = \Phi(\tilde{\mathbf{u}}).$$

   (a) We shall show that this map maps the set $\mathscr{Y}_{(0,T),B}$ into itself for any fixed $B > 0$ and some $T = T(B) > 0$ sufficiently small.

   (b) We shall show that this map is continuous.

   Since the set $\mathscr{Y}_{(0,T),B}$ is a closed convex subset of the space $C([0,T]; L^2(\Omega))$, $\Phi$ admits a fixed point $\mathbf{u} \in \mathscr{Y}_{(0,T),B}$ by the Schauder fixed point theorem.

4. Finally, define the map $A : \mathscr{Y}_{(0,T),B} \to V_{(0,T)}$ by $A(\tilde{\mathbf{u}}) = (\tilde{\beta}, \tilde{\varrho}, \tilde{z})$ and set $A(\mathbf{u}) = (\beta, \mathfrak{r}, \mathfrak{z})$, where $\mathbf{u} \in \mathscr{Y}_{(0,T),B}$ is a fixed point of $\Phi$. Then $(\beta, \mathfrak{r}, \mathfrak{z}, \mathbf{u})$ is a solution of the nonlinear problem (1.7–1.12).

**Step 2:** *Existence, uniqueness and estimates for the problem (4.4–4.7)*
The vector field $\tilde{\mathbf{u}}$ can be extended for $t<0$ in such a way that the new vector field (denoted again $\tilde{\mathbf{u}}$) belongs to $Y_{(-T,T),B}$. It has enough regularity, so that a solution of system (4.4–4.7) can be found by the method of characteristics. Indeed, it can be shown by using contraction mapping argument and then continuation principle (while using the rules of differential calculus of composed mappings in Sobolev spaces, cf. Brezis, Bourgignon [4, Appendix]) that the integral equation for characteristics

$$X(t;x)=x+\int_0^t \tilde{\mathbf{u}}(s;X(s;x))\mathrm{d}s \tag{4.12}$$

admits a unique solution $X\in C([-T,T];W^{3,2}(\Omega))$, $\partial_t X\in Y_{(-T,T)}$ provided $\tilde{\mathbf{u}}\in\mathscr{Y}_{(-T,T),B}$ and there is $C>0$ such that

$$\|X\|_{C([0,T];W^{3,2}(\Omega))}\le C(1+\sqrt{T}\|\tilde{\mathbf{u}}\|_{Y_{(-T,T)}})$$

and

$$\|\partial_t X\|_{Y_{(-T,T)}}\le C\|\tilde{\mathbf{u}}\|_{Y_{(-T,T)}}.$$

Moreover, this solution is such that $X(t,\cdot)$ as well as $X^{-1}(t,\cdot)=X(-t,\cdot)$ are bijections on $\overline{\Omega}$.

It is well known that

$$r(t,x)=r_0(X(-t;x))\exp\Big(-\int_0^t \mathrm{div}\tilde{\mathbf{u}}(s;X(s-t;x))\mathrm{d}s\Big)\in V_{(-T,T)}$$

solves the continuity equation (4.5) resp. (4.6) - if $r_0$ stands for $\mathfrak{r}_0$ resp. for $\mathfrak{z}_0$ - a.e. in $(-T,T)\times\Omega$ (i.e. in particular, in $Q_T$) with the initial data $\mathfrak{r}_0$ resp. $\mathfrak{z}_0$. Likewise,

$$\tilde{\beta}(t,x)=\beta_0(X(-t,x))$$

solves the transport equation (4.4) a.e. in $(-T,T)\times\Omega$ (i.e. in particular, in $Q_T$) with the initial datum $\beta_0$. We can readily derive from the above formulas the following estimates:

$$\forall(t,x)\in\overline{Q}_T,\ \underline{\beta}\le\tilde{\beta}(t,x)\le\overline{\beta}, \tag{4.13}$$

$$\forall(t,x)\in\overline{Q}_T,\ \underline{\mathfrak{r}}_0\exp\Big(-K\sqrt{T}B\Big)\le\tilde{\varrho}(t,x)\le\overline{\mathfrak{r}}_0\exp\Big(K\sqrt{T}B\Big), \tag{4.14}$$

$$\underline{\mathfrak{z}}_0\exp\Big(-K\sqrt{T}B\Big)\le\tilde{z}(t,x)\le\overline{\mathfrak{z}}_0\exp\Big(K\sqrt{T}B\Big)$$

where $\underline{\mathfrak{r}}_0$, $\underline{\mathfrak{z}}_0$ are (strictly positive) infimums of $\mathfrak{r}_0$ and $\mathfrak{z}_0$ over $\Omega$ while $\overline{\mathfrak{r}}_0$ and $\overline{\mathfrak{z}}_0$ are corresponding supremums. Here and hereafter, the positive number $K$ is a universal constant dependent solely on the Sobolev imbeddings (and is, in particular, independent of $T$, $B$ and of the initial data).

Moreover, multiplying equation (4.5) by $\tilde{\varrho}$, $\nabla$(4.5) by $\nabla\tilde{\varrho}$, $\nabla^2$(4.5) by $\nabla^2\tilde{\varrho}$ (and effectuating the same operations with (4.6) and $\tilde{z}$), adding everything and integrating over $(0,\tau)\times\Omega$, we obtain, after a long calculation and an application of the Gronwall lemma (cf. e.g. [27, Lemma 2.4]),

$$\|\tilde{r}\|_{L^\infty(0,T;W^{2,2}(\Omega))}\le K\|r_0\|_{W^{2,2}(\Omega)}\exp\Big(K\sqrt{T}B\Big), \tag{4.15}$$

$$\|\partial_t\tilde{r}\|_{L^\infty(0,T;W^{1,2}(\Omega))} \le KB\|r_0\|_{W^{2,2}(\Omega)}\exp\Big(K\sqrt{T}B\Big),$$

where $\tilde{r}$ stands for $\tilde{\varrho}$ and $\tilde{z}$, respectively, while $r_0$ stands for $\mathfrak{r}_0$ and $\mathfrak{z}_0$, respectively. In the derivation of these estimates, we have again used the rules [4, Appendix]. By the same token,

$$\|\tilde{\beta}\|_{L^\infty(0,T;W^{2,2}(\Omega))} \le K\|\beta_0\|_{W^{2,2}(\Omega)}\exp\Big(K\sqrt{T}B\Big), \tag{4.16}$$

$$\|\partial_t\tilde{\beta}\|_{L^\infty(0,T;W^{1,2}(\Omega))} \le KB\|\beta_0\|_{W^{2,2}(\Omega)}\exp\Big(K\sqrt{T}B\Big).$$

It is rudimentary to show that $(\tilde{\beta}, \tilde{\varrho}, \tilde{z})$ are unique solutions to (4.4–4.7) in class $V_{(0,T)}$.

Seeing the regularity of $f$, $g$, $\tilde{\varrho}$, $\tilde{z}$ and the range of $\tilde{\beta}$ we deduce from (4.4–4.7) that $(\tilde{\beta}, f(\tilde{\beta})\tilde{\varrho}, g(\tilde{\beta})\tilde{z}) \in V_{(0,T)}$ and f$(\tilde{\beta})\tilde{\varrho}$, $g(\tilde{\beta})\tilde{z}$ satisfy equations

$$\partial_t(f(\tilde{\alpha})\tilde{\varrho}) + \operatorname{div}(f(\tilde{\alpha})\tilde{\varrho}\tilde{\mathbf{u}}) = 0, \tag{4.17}$$
$$\partial_t(g(\tilde{\alpha})\tilde{z}) + \operatorname{div}(g(\tilde{\alpha})\tilde{z}\tilde{\mathbf{u}}) = 0 \tag{4.18}$$

with the initial data $f(\alpha_0)\mathfrak{r}_0$, $g(\alpha_0)\mathfrak{z}_0$, respectively. Consequently, in particular, for all $(t,x) \in \overline{Q}_T$,

$$\underline{\mathfrak{r}}\exp\Big(-K\sqrt{T}B\Big) \le f(\tilde{\beta})\tilde{\varrho}(t,x) \le \overline{\mathfrak{r}}\exp\Big(K\sqrt{T}B\Big), \tag{4.19}$$

$$\underline{\mathfrak{z}}\exp\Big(-K\sqrt{T}B\Big) \le g(\tilde{\beta})\tilde{z}(t,x) \le \overline{\mathfrak{z}}\exp\Big(K\sqrt{T}B\Big).$$

We thus observe that there is $\overline{T}_1 = \overline{T}_1(B) > 0$,

$$\mathfrak{q}e^{KB\sqrt{\overline{T}_1}} \le q \tag{4.20}$$

– where $\mathfrak{q}$ is, in order, $\overline{\mathfrak{r}}$, $\overline{\mathfrak{z}}$, $\frac{1}{\underline{\mathfrak{r}}}$, $\frac{1}{\underline{\mathfrak{z}}}$, when $q$ is, in order, $\overline{r}$, $\overline{z}$, $\frac{1}{\underline{r}}$, $\frac{1}{\underline{z}}$– such that for all $T \in (0, \overline{T}_1)$, $(\tilde{\beta}, \tilde{\varrho}, \tilde{z})$ satisfies (4.2) on $\overline{Q}_T$. In the same manner, we get that this triplet satisfies (4.3) on $\overline{Q}_T$. Under these circumstances, we certainly have at least

$$\|(\tilde{\beta}, \tilde{\varrho}, \tilde{z})\|_{V_{(0,T)}} \le CP(I_0, B)\exp\Big(P(I_0, B)\sqrt{T}\Big), \tag{4.21}$$

where

$$I_0 = \|\beta_0, \mathfrak{r}_0, \mathfrak{z}_0\|_{W^{2,2}(\Omega)} + \|\mathbf{u}_0\|_{W^{3,2}(\Omega)}.$$

Here and in the sequel, $P$ is a polynomial, $C > 0$, and $C$ and the coefficients of $P$ are non negative independent of $T$, $B$, $I_0$ (but they may depend, in particular, on $\mu$, $\lambda$, $\Omega$, $\underline{\beta}$, $\overline{\beta}$, $\underline{\mathfrak{r}}$, $\overline{\mathfrak{r}}$, $\underline{\mathfrak{z}}$, $\overline{\mathfrak{z}}$, $\underline{r}$, $\overline{r}$, $\underline{z}$, $\overline{z}$, $\sup_{(r,z)\in[\underline{r},\overline{r}]\times[\underline{z},\overline{z}]} |\nabla_{r,z}p(r,z)|$, $\sup_{\beta\in[\underline{\beta},\overline{\beta}]}(|f'(\beta)| + |g'(\beta)|)$. They can be different in different formulas.

**Step 3**: *Existence, uniqueness and estimates for the equations (4.8–4.10)*
Existence of a unique solution to this system (with coefficients dependent on $(\tilde{\beta}, \tilde{\varrho}, \tilde{z}) \in \mathscr{V}_{(0,T)}$) in the

regularity class $Y_{(0,T)}$ follows from the maximal regularity theory for parabolic equations (cf. e.g. Denk, Hieber, Prüss [8]), provided

$$\mathbf{F} \in W_{(0,T)} \cap C([0,T]; W^{1,2}(\Omega)),\ W_{(0,T)} := \{\mathbf{F} \in L^2([0,T]; W^{1,2}(\Omega)),\ \partial_t \mathbf{F} \in L^2(Q_T)\}.$$

Indeed, one can first solve the problem

$$(\tilde{\varrho} + \tilde{z})\partial_t \mathbf{U} - \mu \Delta \mathbf{U} - (\mu + \lambda)\nabla \mathrm{div} \mathbf{U} + \partial_t(\tilde{\varrho} + \tilde{z})\mathbf{U} = \partial_t \mathbf{F}$$

$$\mathbf{U}|_{(0,T)\times\partial\Omega} = 0,\ \mathbf{U}(0) = \frac{\mathbf{F}(0) + \mu\Delta\mathbf{u}_0 + (\mu+\lambda)\nabla\mathrm{div}\mathbf{u}_0}{(\mathfrak{r}_0 + \mathfrak{z}_0)}.$$

which is (formally) the problem deduced from (4.8–4.11) for $\mathbf{U} = \partial_t \mathbf{u}$ by using [8, Theorem 2.1].

One obtains for this problem estimates by testing the above evolution equation first by $\mathbf{U}$, then by $-\mu\Delta\mathbf{U} - (\mu+\lambda)\nabla\mathrm{div}\mathbf{U}$ and finally by $\partial_t \mathbf{U}$ and employing estimate (4.21). They are resumed in the following formula (see Valli, Zajaczkowski [27, Theorem 2.4] for details),

$$\|\mathbf{U}\|_{L^\infty(0,T;W^{1,2}(\Omega))} + \|\mathbf{U}\|_{L^2(0,T;W^{2,2}(\Omega))} + \|\partial_t \mathbf{U}\|_{L^2(Q_T)}$$
$$\le C\Big(1 + P(I_0,B)\sqrt{T}\Big)\exp\Big(P(I_0,B)Te^{P(I_0,B)\sqrt{T}}\Big)\Big[\|\partial_t \mathbf{F}\|_{L^2(Q_T)} + Q(I_0)\Big].$$

Here and in the sequel $Q$ is a polynomial of one variable with non negative coefficients that are independent of $T$, $B$ and $I_0$ (but may depend on the same variables as the coefficients of the polynomial $P$, cf. (4.21)). It may be different in different formulas.

Then

$$\mathbf{u}(t,\cdot) = \int_0^t \mathbf{U}(s,\cdot)\mathrm{d}s$$

satisfies equation (4.8), i.e.

$$\mu\Delta\mathbf{u} + (\mu+\lambda)\mathbf{u} = \mathbf{U} - \mathbf{F},\ \mathbf{u}|_{\partial\Omega} = 0,$$

to which we may apply the standard elliptic estimates, cf. [17]. Resuming all these calculations, we infer that

$$\|\mathbf{u}\|_{Y_{(0,T)}} \le C\Big(1 + P(I_0,B)\sqrt{T}\Big)\exp\Big(P(I_0,B)Te^{P(I_0,B)\sqrt{T}}\Big)\Big[\|\mathbf{F}\|_{W_{(0,T)}} + Q(I_0)\Big], \tag{4.22}$$

where $\|\mathbf{F}\|_{W_{(0,T)}} = \|\partial_t \mathbf{F}\|_{L^2(Q_T)} + \|\nabla \mathbf{F}\|_{L^2(Q_T)}$.

**Step 4:** *Fixed point of the map* $\Phi$

1. $\Phi$ *maps* $\mathscr{Y}_{(0,T),B}$ *into itself*
   In view of the form of $F$, we find easily (but laboriously) that for all $\tilde{\mathbf{u}} \in \mathscr{Y}_{(0,T),B}$

$$\|\mathbf{F}\|_{W_{(0,T)}} \le P(I_0,B)\sqrt{T}.$$

   Revisiting (4.22) with this information, we get

$$\|\mathbf{u}\|_{Y_{(0,T)}} \le C\Big(1 + P(I_0,B)\sqrt{T}\Big)\exp\Big(P(I_0,B)Te^{P(I_0,B)\sqrt{T}}\Big)\Big[Q(I_0) + P(I_0,B)\sqrt{T}\Big].$$

For given initial data (characterized by $I_0$), we can thus choose $B$ in such a way that

$$B/4 \geq CQ(I_0). \tag{4.23}$$

Choose small $T_1 = T_1(B) > 0$ and $T_2 = T_2(B)$ small enough so that

$$\exp\Big(P(I_0,B)T_1 e^{P(I_0,B)\sqrt{T_1}}\Big) \leq 2$$

and

$$CP(I_0,B)\sqrt{T_2}\Big[1 + Q(I_0) + P(I_0,B)\sqrt{T_2}\Big] \leq \frac{1}{4}B.$$

Once $B$, $T_1$ and $T_2$ are fixed in this way, we take

$$T \leq \min\{T_1, T_2\}. \tag{4.24}$$

With this choice of $B$ and $T$,

$$\Phi(\mathscr{Y}_{(0,T),B}) \subset \mathscr{Y}_{(0,T),B} \text{ and } (\tilde{\beta}, \mathfrak{r}, \mathfrak{z}) \in \mathscr{V}_{(0,T)}. \tag{4.25}$$

2. $\Phi : \mathscr{Y}_{(0,T),B} :\to C(0,T;L^2(\Omega))$ *is continuous on* $\mathscr{Y}_{(0,T),B}$.
Let $\tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2 \in \mathscr{Y}_{(0,T),B}$ for $T$ and $B$ satisfying the hypothesis (4.24). Let $(\tilde{\beta}_i, \tilde{\varrho}_i, \tilde{z}_i)$, $i = 1,2$, be the solutions of the transport equations:

$$\partial_t \tilde{\beta}_i + (\tilde{\mathbf{u}}_i \cdot \nabla)\tilde{\beta}_i = 0, \tag{4.26}$$

$$\partial_t \tilde{\varrho}_i + \mathrm{div}(\tilde{\varrho}_i \tilde{\mathbf{u}}_i) = 0, \tag{4.27}$$

$$\partial_t \tilde{z}_i + \mathrm{div}(\tilde{z}_i \tilde{\mathbf{u}}_i) = 0 \tag{4.28}$$

with the initial conditions

$$\tilde{\beta}_i|_{t=0} = \beta_0, \;\; \tilde{\varrho}_i|_{t=0} = \mathfrak{r}_0, \;\; \tilde{z}_i|_{t=0} = \mathfrak{z}_0. \tag{4.29}$$

Their existence and uniqueness has been established in Step 2. According to (4.25), $(\tilde{\beta}_i, \tilde{\varrho}_i, \tilde{z}_i) \in \mathscr{V}_{(0,T)}$.
Finally let $\mathbf{u}_1, \mathbf{u}_2 \in \mathscr{Y}_{(0,T),B}$ be the solutions to

$$(\tilde{\varrho}_i + \tilde{z}_i)\partial_t \mathbf{u}_i - \mu\Delta\mathbf{u}_i + (\mu+\lambda)\nabla\mathrm{div}\mathbf{u}_i = \mathbf{F}_i \tag{4.30}$$

with the initial conditions

$$\mathbf{u}_i|_{t=0} = \mathbf{u}_0 \tag{4.31}$$

and the boundary conditions

$$\mathbf{u}_i|_{\partial\Omega} = 0, \tag{4.32}$$

where

$$\mathbf{F}_i = -(\tilde{\varrho}_i + \tilde{z}_i)\tilde{\mathbf{u}}_i \cdot \nabla \tilde{\mathbf{u}}_i - \nabla p(f(\tilde{\beta}_i)\tilde{\varrho}_i, g(\tilde{\beta}_i)\tilde{z}_i).$$

Their existence has been established in Step 3.

We shall now write the equations for the differences:

$$\partial_t(\tilde{\beta}_1 - \tilde{\beta}_2) + (\tilde{\mathbf{u}}_1 \cdot \nabla)(\tilde{\beta}_1 - \tilde{\beta}_2) = -(\tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2) \cdot \nabla \tilde{\beta}_2, \tag{4.33}$$

$$\partial_t(\tilde{\varrho}_1 - \tilde{\varrho}_2) + \mathrm{div}((\tilde{\varrho}_1 - \tilde{\varrho}_2)\tilde{\mathbf{u}}_1) = -\mathrm{div}(\tilde{\varrho}_2(\tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2)), \tag{4.34}$$

$$\partial_t(\tilde{z}_1 - \tilde{z}_2) + \mathrm{div}((\tilde{z}_1 - \tilde{z}_2)\tilde{\mathbf{u}}_1) = -\mathrm{div}(\tilde{z}_2(\tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2)), \tag{4.35}$$

$$\begin{aligned}(\tilde{\varrho}_1 + \tilde{z}_1)\partial_t(\mathbf{u}_1 - \mathbf{u}_2) - (\mu\Delta(\mathbf{u}_1 - \mathbf{u}_2) + (\mu + \lambda)\nabla \mathrm{div}(\mathbf{u}_1 - \mathbf{u}_2)) \\ = -(\tilde{\varrho}_1 + \tilde{z}_1 - \tilde{\varrho}_2 - \tilde{z}_2)\partial_t \mathbf{u}_2 + \mathbf{F}_1 - \mathbf{F}_2\end{aligned} \tag{4.36}$$

with the initial conditions

$$\tilde{\alpha}_1 - \tilde{\alpha}_2|_{t=0} = 0,\ \tilde{\varrho}_1 - \tilde{\varrho}_1|_{t=0} = 0,\ z_1 - z_2|_{t=0} = 0,\ \mathbf{u}_1 - \mathbf{u}_2|_{t=0} = 0 \tag{4.37}$$

and the boundary conditions

$$(\mathbf{u}_1 - \mathbf{u}_2)|_{\partial\Omega} = 0. \tag{4.38}$$

Testing (meaning multiplying and then integrating over $(0,\tau) \times \Omega$, $\tau \in (0,T)$) the equation (4.33) by $\tilde{\beta}_1 - \tilde{\beta}_2$, the equation (4.34) by $\tilde{\varrho}_1 - \tilde{\varrho}_2$, the equation (4.35) by $\tilde{z}_1 - \tilde{z}_2$ (in order to evaluate $\tilde{\beta}_1 - \tilde{\beta}_2$ resp. $\tilde{\varrho}_1 - \tilde{\varrho}_2$ resp. $\tilde{z}_1 - \tilde{z}_2$ in terms of the difference $\tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2$), the equation (4.36) scalarly by $\mathbf{u}_1 - \mathbf{u}_2$, using many times Hölder's inequality, Young's inequality and the Poincaré inequality, exploiting the bounds (4.22), (4.25) and the structure of $\mathbf{F}$ (to evaluate $\mathbf{F}_1 - \mathbf{F}_2$ in terms of the differences of $\tilde{\beta}_1 - \tilde{\beta}_2$, $\tilde{\varrho}_1 - \tilde{\varrho}_2$, $\tilde{z}_1 - \tilde{z}_2$, $\tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2$), we obtain, after a straightforward but lengthy calculation

$$\begin{aligned}(\underline{r} + \underline{z})\|(\mathbf{u}_1 - \mathbf{u}_2)(\tau)\|^2_{L^2(\Omega)} + \mu \int_0^\tau \|\nabla \mathbf{u}_1 - \mathbf{u}_2\|^2_{L^2(\Omega)} \mathrm{d}t \\ \leq D \int_0^\tau \|\nabla \mathbf{u}_1 - \nabla \mathbf{u}_2\|^2_{L^2(\Omega)} \mathrm{d}t,\end{aligned}$$

where the number $D$ may depend on the parameters of the problem listed in (4.21) (and also on $T$, $B$, $I_0$, which are now, however, fixed according to (4.23–4.24)) but it is independent of $\tilde{\beta}_i$, $\tilde{\varrho}_i$, $\tilde{z}_i$, $\tilde{\mathbf{u}}_i$. The latter formula implies the continuity of $\Phi$ on $\mathscr{Y}_{(0,T),B}$ in the topology of $C([0,T];L^2(\Omega))$.

3. *Conclusion*
By virtue of the Lions-Aubin lemma, the set $\mathscr{Y}_{(0,T),B}$ is a compact subset of $C([0,T];L^2(\Omega))$. Since it is clearly also convex, we conclude that $\Phi$ admits a fixed point $\mathbf{u} \in \mathscr{Y}_{(0,T),B}$. According to Step 2., the solution $(\beta, \mathfrak{r}, \mathfrak{z})$ of system (4.26–4.28) with $\tilde{\mathbf{u}} = \mathbf{u}$ belongs to $\mathscr{V}_{(0,T)}$. We have thus constructed a (unique) strong solution $(\beta, \mathfrak{r}, \mathfrak{z}, \mathbf{u})$ to problem (1.7–1.12) belonging to $\mathscr{V}_{(0,T)} \times \mathscr{Y}_{(0,T),B}$. Theorem 4.1 is proved.

# 5 Relative energy inequality(with the general test functions)

The goal of this section is to show that any bounded energy weak solution to the Bauer-Nunziato type system (1.7–1.12) is dissipative. We shall start with an auxiliary lemma dealing with the continuity and transport equations, cf. [24, Proposition 6], [22, Proposition 4].

**Lemma 5.1.** *Let $\Omega$ be a bounded Lipschitz domain. Let $r \in C(\overline{I}; L^1(\Omega)) \cap L^\infty(I; L^\gamma(\Omega)) \cap L^2(Q_T)$, $\gamma > 1$, $s \in C(\overline{I}; L^1(\Omega))$, for all $t \in \overline{I}$, $0 \le \underline{s} \le s(t,x) \le \overline{s} < \infty$ for a.a. $x \in \Omega$, $\mathbf{u} \in L^2(I; W_0^{1,2}(\Omega, R^3))$. Suppose that couple $(r, \mathbf{u})$ satisfies continuity equation*

$$\int_\Omega (r\varphi)(\tau,\cdot)\,\mathrm{d}x - \int_\Omega (r\varphi)(0,\cdot)\,\mathrm{d}x = \int_0^\tau \int_\Omega \Big( r\partial_t\varphi + r\mathbf{u}\cdot\nabla\varphi \Big)\,\mathrm{d}x\,\mathrm{d}t \tag{5.1}$$

*for all $\tau \in \overline{I}$ and $\varphi \in C^1_c(\overline{I \times \Omega})$ and couple $(s, \mathbf{u})$ satisfies transport equation*

$$\int_\Omega (s\varphi)(\tau,\cdot)\,\mathrm{d}x - \int_\Omega (s\varphi)(0,\cdot)\,\mathrm{d}x = \int_0^\tau \int_\Omega \Big( s\partial_t\varphi + s\mathbf{u}\cdot\nabla\varphi - s\mathrm{div}\mathbf{u}\varphi \Big)\,\mathrm{d}x\,\mathrm{d}t \tag{5.2}$$

*for all $\tau \in \overline{I}$ and $\varphi \in C^1_c(\overline{I \times \Omega})$. Then for any $b \in C^1([\underline{s}, \overline{s}])$, $rb(s) \in C(\overline{I}; L^1(\Omega))$ and couple $(rb(s), \mathbf{u})$ satisfies continuity equation*

$$\int_\Omega (rb(s)\varphi)(\tau,\cdot)\,\mathrm{d}x - \int_\Omega (rb(s)\varphi)(0,\cdot)\,\mathrm{d}x = \int_0^\tau \int_\Omega \Big( rb(s)\partial_t\varphi + rb(s)\mathbf{u}\cdot\nabla\varphi \Big)\,\mathrm{d}x\,\mathrm{d}t \tag{5.3}$$

*for all $\tau \in \overline{I}$ and $\varphi \in C^1_c(\overline{I \times \Omega})$*

With Lemma 5.1 at hand, we can prove the main theorem of this section.

**Theorem 5.2.** *Let $\Omega$ be a bounded Lipschitz domain. Let $\mathcal{O} \subset (0,\infty)^2$ be an open set and $0 \le \underline{\alpha} < \overline{\alpha} \le 1$. Let $(\alpha, \varrho, z, \mathbf{u})$ with $(\varrho, z) \in L^2(Q_T)$ be a bounded energy weak solution to the problem (1.7–1.12) with the densities ranging in $\overline{\mathcal{O}}$ and the concentration in $[\underline{\alpha}, \overline{\alpha}]$ according to the Definition 2.1, where*

$$p \in C(\overline{\mathcal{O}}),\ H \in C(\overline{\mathcal{O}}) \cap C^2(\mathcal{O}),\ f, g \in C^1(0,1) \cap C^1[\underline{\alpha}, \overline{\alpha}].$$

*Then it is a dissipative solution of the same system. In particular, it satisfies the relative energy inequality (2.8) with any set of test functions $(\beta, \mathfrak{r}, \mathfrak{z}, \mathbf{U})$ in the class (2.9).*

**Remark 5.1**
As in the "mono-fluid" case, one can show by a density argument that the test functions $(\beta, \mathfrak{r}, \mathfrak{z}, \mathbf{U})$ can be taken in a slightly broader class than (2.9), namely

$$(\beta, \mathfrak{r}, \mathfrak{z}, \mathbf{U}) \in C([0,T] \times \overline{\Omega}),\ \partial_t\beta \in L^2(Q_T),\ \nabla\beta \in C(\overline{Q_T}),\ \partial_t(\mathfrak{r}, \mathfrak{z}, \mathbf{U}),\ \nabla(\mathfrak{r}, \mathfrak{z}, \mathbf{U}) \in L^2(0,T; C(\overline{\Omega})), \tag{5.4}$$

$$\underline{\alpha} \le \beta \le \overline{\alpha},\ (\mathfrak{r}(t,x), \mathfrak{z}(t,x)) \in \overline{\mathcal{O}},\ \mathbf{U}|_{\partial\Omega} = 0.$$

The rest of this section is devoted to the proof of Theorem 5.2

**Proof of Theorem 5.2**

Throughout this proof $(\beta, \mathfrak{r}, \mathfrak{z}, \mathbf{U})$ is any quadruple in the regularity class (2.9).

If we take in $(2.2)_{r=\varrho}$ and in $(2.2)_{r=z}$, the function $\varphi = \frac{|\mathbf{U}|^2}{2}$ as the test function, we obtain the identity

$$\int_\Omega (\varrho + z)\frac{|\mathbf{U}|^2}{2}dx\Big|_0^\tau = \int_0^\tau \int_\Omega (\varrho + z)\mathbf{U}\cdot\Big(\partial_t \mathbf{U} + (\mathbf{u}\cdot\nabla)\mathbf{U}\Big)dxdt. \qquad (5.5)$$

The equation (2.4) with the test function $\varphi = \mathbf{U}$, reads

$$-\int_\Omega (\varrho + z)\mathbf{u}\cdot\mathbf{U}dx\Big|_0^\tau = -\int_0^\tau \int_\Omega \Big[(\varrho + z)\mathbf{u}\cdot\Big(\partial_t \mathbf{U} + (\mathbf{u}\cdot\nabla)\mathbf{U}\Big)$$
$$+p(f(\alpha)\varrho, g(\alpha)z)\mathrm{div}\mathbf{U} - \mathbb{S}(\nabla\mathbf{u}) : \nabla\mathbf{U}\Big]dxdt. \qquad (5.6)$$

In view of Lemma 5.1, the equation $(2.2)_{r=\varrho}$ with the test function $\varphi = \partial_R H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})$ yields,

$$-\int_\Omega \varrho f(\alpha)\partial_R H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})dx\Big|_0^\tau = -\int_0^\tau \int_\Omega f(\alpha)\varrho\Big(\partial_t + (\mathbf{u}\cdot\nabla)\Big)\partial_R H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})dxdt, \qquad (5.7)$$

while, by the same token, the equation $(2.2)_{r=z}$ with the test function $\varphi = \partial_Z H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})$ gives, in particular,

$$-\int_\Omega zg(\alpha)\partial_Z H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})dx\Big|_0^\tau = -\int_0^\tau \int_\Omega g(\alpha)z\Big(\partial_t + (\mathbf{u}\cdot\nabla)\Big)\partial_Z H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})dxdt. \qquad (5.8)$$

We deduce from the equation (2.5) written in the form

$$f(\beta)\mathfrak{r}\partial_R H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z}) + g(\beta)\mathfrak{z}\partial_Z H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z}) - H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z}) = p(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z}),$$

the integral identity

$$\int_\Omega \Big(f(\beta)\mathfrak{r}\partial_R H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z}) + g(\beta)\mathfrak{z}\partial_Z H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z}) - H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})\Big)dx\Big|_0^\tau \qquad (5.9)$$
$$= \int_0^\tau \int_\Omega \partial_t p(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})dxdt.$$

Summing up the energy inequality (2.6) and the identities (5.5–5.9), we arrive at the inequality

$$\int_\Omega \Big[\frac{(\varrho + z)}{2}|\mathbf{u} - \mathbf{U}|^2 + H(f(\alpha)\varrho, g(\alpha)z) - H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})$$
$$+(f(\beta)\mathfrak{r} - f(\alpha)\varrho)\partial_R H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z}) + (g(\beta)\mathfrak{z} - g(\alpha)z)\partial_Z H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})\Big]dx\Big|_0^\tau$$

$$
\begin{aligned}
&+\int_0^\tau\int_\Omega \mathbb{S}(\nabla\mathbf{u}):\nabla(\mathbf{u}-\mathbf{U})dxdt\\
\le&\int_0^\tau\int_\Omega(\varrho+z)\Big(\partial_t\mathbf{U}+\mathbf{u}\cdot\nabla\mathbf{U}\Big)\cdot(\mathbf{U}-\mathbf{u})dxdt\\
&-\int_0^\tau\int_\Omega p(f(\alpha)\varrho,g(\alpha)z)\mathrm{div}\mathbf{U}dxdt\\
&-\int_0^\tau\int_\Omega f(\alpha)\varrho\Big(\partial_t+\mathbf{u}\cdot\nabla\Big)\partial_R H(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})dxdt\\
&-\int_0^\tau\int_\Omega g(\alpha)z\Big(\partial_t+\mathbf{u}\cdot\nabla\Big)\partial_Z H(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})dxdt\\
&+\int_0^\tau\int_\Omega \partial_t p(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})dxdt. \qquad (5.10)
\end{aligned}
$$

Adding to the both sides of (5.10) the term $-\int_0^\tau\int_\Omega \mathbb{S}(\nabla\mathbf{U}):\nabla(\mathbf{u}-\mathbf{U})dxdt$ one gets

$$
\begin{aligned}
&\int_\Omega\Big[\frac{(\varrho+z)}{2}|\mathbf{u}-\mathbf{U}|^2+H(f(\alpha)\varrho,g(\alpha)z)-H(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})\\
&+(f(\beta)\mathfrak{r}-f(\alpha)\varrho)\partial_R H(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})+(g(\beta)\mathfrak{z}-g(\alpha)\mathfrak{z})\partial_Z H(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})\Big]dx\Big|_0^\tau\\
&+\int_0^\tau\int_\Omega \mathbb{S}(\nabla\mathbf{u}-\nabla\mathbf{U}):\nabla(\mathbf{u}-\mathbf{U})dxdt\\
\le&\int_0^\tau\int_\Omega\Big[(\varrho+z)(\mathbf{U}-\mathbf{u})\cdot\Big(\partial_t\mathbf{U}+\mathbf{u}\cdot\nabla\mathbf{U}\Big)+\mathbb{S}(\nabla\mathbf{U}):\nabla(\mathbf{U}-\mathbf{u})\Big]dxdt\\
&-\int_0^\tau\int_\Omega p(f(\alpha)\varrho,g(\alpha)z)\mathrm{div}\mathbf{U}dxdt\\
&-\int_0^\tau\int_\Omega f(\alpha)\varrho\Big(\partial_t+\mathbf{u}\cdot\nabla\Big)\partial_R H(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})dxdt\\
&-\int_0^\tau\int_\Omega g(\alpha)z\Big(\partial_t+\mathbf{u}\cdot\nabla\Big)\partial_Z H(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})dxdt\\
&+\int_0^\tau\int_\Omega \partial_t p(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})dxdt. \qquad (5.11)
\end{aligned}
$$

Observing that

$$
\partial_t H(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})=\partial_R H(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})\partial_t(f(\beta)\mathfrak{r})+\partial_Z H(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})\partial_t(g(\beta)\mathfrak{z})
$$

we verify that

$$
\begin{aligned}
\partial_t p(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})&=\partial_t\Big[-H((f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})+f(\beta)\mathfrak{r}\partial_R H(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})+g(\beta)\mathfrak{z}\partial_Z H(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})\Big]\\
&=f(\beta)\mathfrak{r}\partial_t\partial_R H(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})+g(\beta)\mathfrak{z}\partial_t\partial_Z H(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z}). \qquad (5.12)
\end{aligned}
$$

Likewise, observing that

$$\nabla H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z}) = \partial_R H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})\nabla(f(\beta)\mathfrak{r}) + \partial_Z H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})\nabla(g(\beta)\mathfrak{z}),$$

we verify the identity

$$\begin{aligned}
&- \int_\Omega p(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})\mathrm{div}\mathbf{U}dx \\
&= \int_\Omega \Big[H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z}) - f(\beta)\mathfrak{r}\partial_R H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z}) - g(\beta)\mathfrak{z}\partial_Z H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})\Big]\mathrm{div}\mathbf{U}dx \\
&= -\int_\Omega \mathbf{U}\cdot\nabla\Big(H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z}) - f(\beta)\mathfrak{r}\partial_R H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z}) - g(\beta)\mathfrak{z}\partial_Z H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})\Big)dx \\
&= \int_\Omega \Big(f(\beta)\mathfrak{r}\mathbf{U}\cdot\nabla\partial_R H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z}) + g(\beta)\mathfrak{z}\mathbf{U}\cdot\nabla\partial_Z H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})\Big)dx. \qquad (5.13)
\end{aligned}$$

Employing (5.12) and (5.13) we finally arrive at

$$\begin{aligned}
&- \int_\Omega p(f(\alpha)\varrho, g(\alpha)z)\mathrm{div}\mathbf{U}dx - \int_\Omega f(\alpha)\varrho\Big(\partial_t + \mathbf{u}\cdot\nabla\Big)\partial_R H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})dx \\
&- \int_\Omega g(\alpha)z\Big(\partial_t + \mathbf{u}\cdot\nabla\Big)\partial_Z H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})dx + \int_\Omega \partial_t p(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})dx \\
&= \int_\Omega \Big(f(\beta)\mathfrak{r} - f(\alpha)\varrho\Big)\partial_t\partial_R H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z}) + \Big(g(\beta)\mathfrak{z} - g(\alpha)z\Big)\partial_t\partial_Z H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})dx \\
&+ \int_\Omega \Big(p(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z}) - p(f(\alpha)\varrho, g(\alpha)z)\Big)\mathrm{div}\mathbf{U}dx \\
&+ \int_\Omega \Big(f(\beta)\mathfrak{r}\mathbf{U} - f(\alpha)\varrho\mathbf{u}\Big)\cdot\nabla\partial_R H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z}) + \Big(g(\beta)\mathfrak{z}\mathbf{U} - g(\alpha)z\mathbf{u}\Big)\cdot\nabla\partial_Z H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})dx. \qquad (5.14)
\end{aligned}$$

With this identity at hand, the right hand side of (5.11) can be rewritten as follows

$$\begin{aligned}
&\int_0^\tau \int_\Omega \Big[(\varrho + z)(\mathbf{U} - \mathbf{u})\cdot\Big(\partial_t\mathbf{U} + \mathbf{u}\cdot\nabla\mathbf{U}\Big) + \mathbb{S}(\nabla\mathbf{U}) : \nabla(\mathbf{U} - \mathbf{u})\Big]dxdt \\
&+ \int_0^\tau \int_\Omega \Big(f(\beta)\mathfrak{r} - f(\alpha)\varrho\Big)\partial_t\partial_R H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})dxdt \\
&+ \int_0^\tau \int_\Omega \Big(g(\beta)\mathfrak{z} - g(\alpha)z\Big)\partial_t\partial_Z H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})dxdt \\
&+ \int_0^\tau \int_\Omega \Big(p(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z}) - p(f(\alpha)\varrho, g(\alpha)z)\Big)\mathrm{div}\mathbf{U}dxdt \\
&+ \int_0^\tau \int_\Omega \Big(f(\beta)\mathfrak{r}\mathbf{U} - f(\alpha)\varrho\mathbf{u}\Big)\cdot\nabla\partial_R H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})dxdt \\
&+ \int_0^\tau \int_\Omega \Big(g(\beta)\mathfrak{z}\mathbf{U} - g(\alpha)z\mathbf{u}\Big)\cdot\nabla\partial_Z H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})dxdt. \qquad (5.15)
\end{aligned}$$

This completes the proof of Theorem 5.2.

# 6 Relative energy inequality with the strong solution

We employ in the relative energy inequality (2.8) the test functions $(\beta, \mathfrak{r}, \mathfrak{z}, \mathbf{U})$, where $(\beta, \mathfrak{r}, \mathfrak{z}, \mathbf{U})$ is a strong solution of system (1.7–1.12) in the class (2.9) with the initial data $(\beta_0, \mathfrak{r}_0, \mathfrak{z}_0, \mathbf{U}_0)$. In this case, the remainder in the relative energy inequality becomes quadratic in the difference of the weak and the strong solution. This property is formulated rigorously in the following theorem:

**Theorem 6.1.** *Let all assumptions of Theorem 5.2 be satisfied and suppose moreover that* $(\beta, \mathfrak{r}, \mathfrak{z}, \mathbf{U})$ *is a strong solution to the one velocity Baer-Nunziato type system (1.7–1.12) in the class (2.9) emanating from the initial data* $(\beta_0, \mathfrak{r}_0, \mathfrak{z}_0, \mathbf{U}_0)$. *Then the remainder in the relative energy inequality (2.8) takes the form:*

$$
\begin{aligned}
&\mathscr{R}_{\alpha,\beta}(\varrho, z, \mathbf{u}|\mathfrak{r}, \mathfrak{z}, \beta, \mathbf{U}) \\
&= \int_\Omega (\mathbf{U}-\mathbf{u}) \cdot \Big[(\varrho + z - \mathfrak{r} - \mathfrak{z})\partial_t \mathbf{U} + \Big((\varrho+z)\mathbf{u} - (\mathfrak{r}+\mathfrak{z})\mathbf{U}\Big)\cdot \nabla \mathbf{U}\Big] dx \\
&+ \int_\Omega \Big(p(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z}) - p(f(\alpha)\varrho, g(\alpha)z) - \partial_R p(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})(f(\beta)\mathfrak{r} - f(\alpha)\varrho) \\
&\qquad - \partial_Z p(f(\beta)\mathfrak{r}, g(\beta\mathfrak{z})(g(\beta)\mathfrak{z} - g(\alpha)z)\Big) \mathrm{div}\mathbf{U} dx \\
&+ \int_\Omega \Big(f(\alpha)\varrho - f(\beta)\mathfrak{r}\Big)(\mathbf{U}-\mathbf{u}) \cdot \nabla \partial_R H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z}) dx \\
&+ \int_\Omega \Big(g(\alpha)z - g(\beta)\mathfrak{z}\Big)(\mathbf{U}-\mathbf{u}) \cdot \nabla \partial_Z H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z}) dx. \qquad (6.1)
\end{aligned}
$$

**Remark 6.1**
The strong solution in Theorem 6.1 can be taken in the class (5.4) which is slightly broader than (2.9), cf. Remark 5.1.

The rest of this section will be devoted to the proof of Theorem 6.1.
**Proof of Theorem 6.1**

Recall that

$$
\begin{aligned}
&\mathscr{R}_{\alpha,\beta}(\varrho, z, \mathbf{u}|\mathfrak{r}, \mathfrak{z}, \mathbf{U}) \\
&= \int_\Omega (\mathbf{U}-\mathbf{u}) \cdot \Big[(\varrho+z)\Big(\partial_t \mathbf{U} + (\mathbf{u}\cdot\nabla)\mathbf{U}\Big) - \mathrm{div}\mathbb{S}(\nabla \mathbf{U})\Big] dx \\
&+ \int_\Omega \Big(f(\beta)\mathfrak{r} - f(\alpha)\varrho\Big)\partial_t \partial_R H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z}) dx \\
&+ \int_\Omega \Big(g(\beta)\mathfrak{z} - g(\alpha)z\Big)\partial_t \partial_Z H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z}) dx \\
&+ \int_\Omega \Big(p(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z}) - p(f(\alpha)\varrho, g(\alpha)z)\Big)\mathrm{div}\mathbf{U} dx
\end{aligned}
$$

$$+\int_\Omega \Big(f(\beta)\mathfrak{r}\mathbf{U} - f(\alpha)\varrho\mathbf{u}\Big)\cdot\nabla\partial_R H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})dx$$
$$+\int_\Omega \Big(g(\beta)\mathfrak{z}\mathbf{U} - g(\alpha)z\mathbf{u}\Big)\cdot\nabla\partial_Z H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})dx$$
$$= I + II + III + IV + V + VI, \tag{6.2}$$

where we have used (2.8) and the integration by parts which gives the identity $\int_\Omega(\mathbf{U}-\mathbf{u})\cdot\mathrm{div}\mathbb{S}(\nabla\mathbf{U})dx$ $= -\int_\Omega \mathbb{S}(\nabla\mathbf{U}) : \nabla(\mathbf{U}-\mathbf{u})dx$. Indeed, in view of (2.9), $\mathrm{div}\mathbb{S}(\nabla\mathbf{U}) \in C([0,T]\times\overline{\Omega})$ by virtue of (1.10).

Since $(\mathfrak{r}+\mathfrak{z})(\partial_t\mathbf{U} + \mathbf{U}\cdot\nabla\mathbf{U}) - \mathrm{div}\mathbb{S}(\nabla\mathbf{U}) + \nabla p(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z}) = 0$, $I$ can be rewritten as

$$I = \int_\Omega (\mathbf{U}-\mathbf{u})\cdot\Big[(\varrho+z)\Big(\partial_t\mathbf{U} + \mathbf{u}\cdot\nabla\mathbf{U}\Big)$$
$$-(\mathfrak{r}+\mathfrak{z})(\partial_t\mathbf{U} + \mathbf{U}\cdot\nabla\mathbf{U}) - \nabla p(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})\Big]dx$$
$$= \int_\Omega (\mathbf{U}-\mathbf{u})\cdot\Big[(\varrho+z-\mathfrak{r}-\mathfrak{z})\partial_t\mathbf{U} + \Big((\varrho+z)\mathbf{u} - (\mathfrak{r}+\mathfrak{z})\mathbf{U}\Big)\cdot\nabla\mathbf{U}\Big]dx$$
$$-\int_\Omega (\mathbf{U}-\mathbf{u})\cdot\nabla p(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})dx = I_1 + I_2. \tag{6.3}$$

Since $\partial_t(f(\beta)\mathfrak{r}) + \mathrm{div}(f(\beta)\mathfrak{r}\mathbf{U}) = 0$ and $\partial_t(g(\beta)\mathfrak{z}) + \mathrm{div}(g(\beta)\mathfrak{z}\mathbf{U}) = 0$, we have

$$\partial_t\partial_R H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z}) = -\mathbf{U}\cdot\nabla\partial_R H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z}) - \partial_R p(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})\mathrm{div}\mathbf{U}$$

and

$$\partial_t\partial_Z H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z}) = -\mathbf{U}\cdot\nabla\partial_Z H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z}) - \partial_Z p(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})\mathrm{div}\mathbf{U}.$$

Here we used the fact that

$$R\partial_R^2 H(R,Z) + Z\partial_Z\partial_R H(R,Z) = \partial_R p(R,Z)$$

and

$$R\partial_R\partial_Z H(R,Z) + Z\partial_Z^2 H(R,Z) = \partial_Z p(R,Z).$$

Hence $II$ and $III$ can be rewritten as follows

$$II = -\int_\Omega \Big(f(\beta)\mathfrak{r} - f(\alpha)\varrho\Big)\mathbf{U}\cdot\nabla\partial_R H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})dx$$
$$-\int_\Omega \Big(f(\beta)\mathfrak{r} - f(\alpha)\varrho\Big)\partial_R p(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})\mathrm{div}\mathbf{U}dx = II_1 + II_2 \tag{6.4}$$

and

$$III = -\int_\Omega \Big(g(\beta)\mathfrak{z} - g(\alpha)z\Big)\mathbf{U}\cdot\nabla\partial_Z H(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})dx$$

$$-\int_\Omega \Big(g(\beta)\mathfrak{z}-g(\alpha)z\Big)\partial_Z p(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})\mathrm{div}\mathbf{U}dx = III_1+III_2. \tag{6.5}$$

Combining $IV$, $II_2$ and $III_2$, we have

$$IV+II_2+III_2=\int_\Omega \Big(p(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})-p(f(\alpha)\varrho,g(\alpha)z)$$
$$-\partial_R p(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})(f(\beta)\mathfrak{r}-f(\alpha)\varrho)-\partial_Z p(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})(g(\beta)\mathfrak{z}-g(\alpha)z)\Big)\mathrm{div}\mathbf{U}dx. \tag{6.6}$$

Observe that

$$V+II_1=\int_\Omega f(\alpha)\varrho(\mathbf{U}-\mathbf{u})\cdot\nabla\partial_R H(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})dx \tag{6.7}$$

and

$$VI+III_1=\int_\Omega g(\alpha)z(\mathbf{U}-\mathbf{u})\cdot\nabla\partial_Z H(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})dx. \tag{6.8}$$

Since

$$\nabla H(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})=\partial_R H(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})\nabla(f(\beta)\mathfrak{r})+\partial_Z H(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})\nabla(g(\beta)\mathfrak{z}),$$

we get

$$\nabla p(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})=\nabla\Big(-H(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})+f(\beta)\mathfrak{r}\partial_R H(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})+g(\beta)\mathfrak{z}\partial_Z H(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})\Big)$$
$$=f(\beta)\mathfrak{r}\nabla\partial_R H(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})+g(\beta)\mathfrak{z}\nabla\partial_Z H(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z}).$$

Hence, we have

$$V+II_1+VI+III_1+I_2=\int_\Omega \Big(f(\alpha)\varrho-f(\beta)\mathfrak{r}\Big)(\mathbf{U}-\mathbf{u})\cdot\nabla\partial_R H(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})dx$$
$$+\int_\Omega \Big(g(\alpha)z-g(\beta)\mathfrak{z}\Big)(\mathbf{U}-\mathbf{u})\cdot\nabla\partial_Z H(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})dx. \tag{6.9}$$

From $I_1$ in (6.3), (6.6) and (6.9) we conclude that

$$\mathscr{R}_{\alpha,\beta}(\varrho,z,\mathbf{u}|\mathfrak{r},\tilde{z},\mathbf{U})$$
$$=\int_\Omega(\mathbf{U}-\mathbf{u})\cdot\Big[(\varrho+z-\mathfrak{r}-\mathfrak{z})\partial_t\mathbf{U}+\Big((\varrho+z)\mathbf{u}-(\mathfrak{r}+\mathfrak{z})\mathbf{U}\Big)\cdot\nabla\mathbf{U}\Big]dx$$
$$+\int_\Omega\Big(p(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})-p(f(\alpha)\varrho,g(\alpha)z)-\partial_R p(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})(f(\beta)\mathfrak{r}-f(\alpha)\varrho)$$
$$-\partial_Z p(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})(g(\beta)\mathfrak{z}-g(\alpha)z)\Big)\mathrm{div}\mathbf{U}dx$$
$$+\int_\Omega\Big(f(\alpha)\varrho-f(\beta)\mathfrak{r}\Big)(\mathbf{U}-\mathbf{u})\cdot\nabla\partial_R H(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})dx$$
$$+\int_\Omega\Big(g(\alpha)z-g(\beta)\mathfrak{z}\Big)(\mathbf{U}-\mathbf{u})\cdot\nabla\partial_Z H(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})dx. \tag{6.10}$$

Theorem 6.1 is proved.

# 7 The weak-strong uniqueness

We start with the following simple algebraic lemma dealing with functions of two variables which will be systematically used in the weak-strong uniqueness proof.

**Lemma 7.1.** *Let $L, K \subset \mathcal{O}$ be two compact sets such that $K \subset \mathrm{int} L$, where $\mathcal{O}$ is a convex subset of $(0,\infty)^2$. Let $H \in C(\overline{\mathcal{O}}) \cap C^2(\mathcal{O})$ have a (strictly) positive Hessian matrix. Suppose that there are numbers $C > 0$ and $\overline{R} > 1$ such that*

$$\forall (R,Z) \in \mathcal{O},\ |R| + |Z| \geq \overline{R},\ R^\xi + Z^\xi + p(R,Z) \leq CH(R,Z) \text{ with some } \xi > 1. \tag{7.1}$$

*Then there exists a positive number $c = c(K, L, \delta, C, \overline{R})$ such that for all $(\mathfrak{R}, \mathfrak{Z}) \in K$ and all $(R,Z) \in \overline{\mathcal{O}}$,*

$$\Big[\Big((R-\mathfrak{R})^2 + (Z-\mathfrak{Z})^2\Big)1_L(R,Z) + \Big(1 + R^\xi + Z^\xi + p(R,Z)\Big)1_{\overline{\mathcal{O}}\setminus L}(R,Z)\Big] \leq cE(R,Z\,|\,\mathfrak{R},\mathfrak{Z}).$$

*Recall that $E$ is defined in (2.7).*

**Proof of Lemma 7.1**
We have for the Hessian of $H$,

$$\forall \mathbf{h} \in \mathbb{R}^2,\ \inf_{(\mathfrak{R},\mathfrak{Z})\in L} \mathbf{h}^T D^2 H(\mathfrak{R},\mathfrak{Z})\mathbf{h} \geq \underline{c}|\mathbf{h}|^2 \text{ with some } \underline{c} > 0.$$

Consequently, the inequality

$$\Big((R-\mathfrak{R})^2 + (Z-\mathfrak{Z})^2\Big)1_L(R,Z) \leq cE(R,Z\,|\,\mathfrak{R},\mathfrak{Z})$$

is a consequence of (2.7) and the second order Taylor formula.

Due to the strict convexity of $H$ the map $(R,Z) \mapsto E(R,Z|\mathfrak{R},\mathfrak{Z})$ has in $\mathcal{O}$ a unique global minimum in the point $(\mathfrak{R},\mathfrak{Z}) \in K$, which is equal to 0. Consequently, there is $\underline{c} > 0$ such that

$$\inf_{(R,Z)\in\mathcal{O}\setminus L, (\mathfrak{R},\mathfrak{Z})\in K} E(R,Z|\mathfrak{R},\mathfrak{Z}) \geq \underline{c} \tag{7.2}$$

so that we have for all $(\mathfrak{R},\mathfrak{Z}) \in K$

$$\max_{(R,Z)\in\overline{\mathcal{O}}\cap[0,\overline{R}]^2} p(R,Z) := \overline{p} \leq \frac{\overline{p}}{\underline{c}} E(R,Z|\mathfrak{R},\mathfrak{Z}).$$

This yields the inequality

$$1_{\overline{\mathcal{O}}\setminus L}\, p(R,Z) \leq cE(R,Z\,|\,\mathfrak{R},\mathfrak{Z}). \tag{7.3}$$

Finally, by virtue of the assumption (7.1), we have for all $R + Z \geq \overline{R}$,

$$p(R,Z) + R^\xi + Z^\xi \leq CE(R,Z|\mathfrak{R},\mathfrak{Z}) + C\Big(\partial_R H(\mathfrak{R},\mathfrak{Z})(R-\mathfrak{R})$$

$$+\partial_Z H(\mathfrak{R},\mathfrak{Z})(Z-\mathfrak{Z}) + H(\mathfrak{R},\mathfrak{Z})\Big)$$
$$\leq CE(R,Z|\mathfrak{R},\mathfrak{Z}) + A(R+Z) + B.$$

where we have denoted

$$A := C \max_{(\mathfrak{R},\mathfrak{Z})\in K} \Big(\Big|\partial_R H(\mathfrak{R},\mathfrak{Z})\Big| + \Big|\partial_R H(\mathfrak{R},\mathfrak{Z})\Big|\Big)$$

$$B := C \max_{(\mathfrak{R},\mathfrak{Z})\in K} \Big(\Big|\partial_R H(\mathfrak{R},\mathfrak{Z})\mathfrak{R}\Big| + \Big|\partial_Z H(\mathfrak{R},\mathfrak{Z})\mathfrak{Z}\Big| + \Big|H(\mathfrak{R},\mathfrak{Z})\Big|\Big).$$

Now we may employ at the right hand side of the last inequality the Young inequality in order to "absorb" the term $A(R+Z)$ at the left hand side, and then use estimate (7.2). This yields, in particular,

$$\Big(p(R,Z) + R^\xi + Z^\xi\Big)1_{\overline{\mathscr{O}}\setminus L} \leq CE(R,Z|\mathfrak{R},\mathfrak{Z})$$

and finishes the proof of Lemma 7.1.

Next theorem compares a weak solution on interval $(0,T)$ emanating from the initial data $(\alpha_0,\varrho_0,z_0,\mathbf{u}_0)$ with a strong solution on the same interval emanating from the initial data $(\beta_0,\mathfrak{r}_0,\mathfrak{z}_0,\mathbf{U}_0)$ (provided it exists on interval $[0,T)$). It yields, in particular, the weak-strong uniqueness principle for the one velocity Baer-Nunziato type system (1.7–1.12).

**Theorem 7.2.** *Let $\Omega$ be a bounded Lipschitz domain. Let $\mathscr{O}$ be an open convex subset of $(0,\infty)^2$, $0\leq\underline{\alpha}<\overline{\alpha}\leq 1$. Assume that*

$$f,g\in C^1(0,1)\cap C^1([\underline{\alpha},\overline{\alpha}]),\ p\in C(\overline{\mathscr{O}})\cap C^2(\mathscr{O}),\ H\in C(\overline{\mathscr{O}})\cap C^2(\mathscr{O})$$

*where the Hesian matrix of $H$ is (strictly) positive on $\mathscr{O}$ and satisfies relation (7.1), and $f,g$ are non-vanishing on $[\underline{\alpha},\overline{\alpha}]$.*

*Let $(\alpha,\varrho,z,\mathbf{u})$ be a weak solution with densities ranging in $\overline{\mathscr{O}}$ and concentration in $[\underline{\alpha},\overline{\alpha}]$ according to Definition 2.1 emanating from initial data*

$$\underline{\alpha}\leq\alpha_0\leq\overline{\alpha},\ (f(\alpha_0)\varrho_0, g(\alpha_0)z_0)\in\overline{\mathscr{O}},$$
$$\varrho_0\in L^\gamma(\Omega),\ \gamma>1,\ \mathbf{u}_0\in L^1(\Omega),\ (\varrho_0+z_0)\mathbf{u}_0^2\in L^1(\Omega).$$

*Let $(\beta,\mathfrak{r},\mathfrak{z},\mathbf{U})$ be a strong solution of the same system in the class (5.4) emanating from initial data $(\beta_0,\mathfrak{r}_0,\mathfrak{z}_0,\mathbf{U}_0) = (\beta(0),\mathfrak{r}(0),\mathfrak{z}(0),\mathbf{U}(0))$. Then there exists a number $C>0$ independent of the weak solution and its initial data (but dependent possibly on the strong solution through the norms, parameters and sets indicated in (5.4)), such that for a.a. $\tau\in(0,T)$,*

$$\mathscr{E}\Big(\alpha,\varrho,z,\mathbf{u}\,\Big|\,\beta,\mathfrak{r},\mathfrak{z},\mathbf{U}\Big)(\tau) + \frac{1}{2}\int_0^\tau\int_\Omega \mathbb{S}(\nabla(\mathbf{u}-\mathbf{U})):\nabla(\mathbf{u}-\mathbf{U})\ \mathrm{d}x\mathrm{d}t \leq C\mathscr{E}\Big(\alpha_0,\varrho_0,z_0,\mathbf{u}_0\,\Big|\,\beta_0,\mathfrak{r}_0,\mathfrak{z}_0,\mathbf{U}_0\Big),$$

*where*

$$\mathscr{E}\Big(\alpha,\varrho,z,\mathbf{u}\,\Big|\,\beta,\mathfrak{r},\mathfrak{z},\mathbf{U}\Big) := \int_\Omega\Big(\frac{1}{2}|\alpha-\beta|^2 + \frac{1}{2}(\varrho+z)|\mathbf{u}-\mathbf{U}|^2 + E(f(\alpha)\varrho, g(\alpha)z\,|\,f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})\Big)(\tau,x)\ \mathrm{d}x$$

*In particular, if $(\beta_0,\mathfrak{r}_0,\mathfrak{z}_0,\mathbf{U}_0) = (\alpha_0,\varrho_0,z_0,\mathbf{u}_0)$ then $(\alpha,\varrho,z,\mathbf{u}) = (\beta,\mathfrak{r},\mathfrak{z},\mathbf{U})$.*

**Remark 7.1**

1. The assumptions of Theorem 7.2 are not void. Indeed, there is at least one setting (characterized by $\mathscr{O}$) in which the weak solutions exist on an arbitrary large interval $I$ and the strong solutions exist at least on a short time interval $I_*$. Indeed: The weak solutions satisfying the assumptions of Theorem 7.2 on an arbirary large interval $I = (0, T)$ with the convex set $\mathscr{O} = \mathscr{O}_{\underline{a},\overline{a}}$, $0 < \underline{a} < \overline{a}$ defined in (3.1) have been constructed [22, Theorem 1] provided $0 < \underline{\alpha} < \overline{\alpha} < 1$ and $f, g$ are strictly monotone non vanishing on $(0, 1)$. Their existence is recalled in Theorem 3.1 (and Remark 3.1). The strong solutions satisfying the assumptions of Theorem 7.2 have been constructed on a short time interval $I_* = [0, T_*)$ in Theorem 4.1, provided the domain $\Omega$ is of class $C^3$ and the initial data are sufficiently regular and verify the compatibility conditions at the boundary. Indeed, if we take in Theorem 4.1 $\underline{\beta} = \underline{\alpha}$, $\overline{\beta} = \overline{\alpha}$ and $0 < \underline{b} < \overline{b} < \overline{a}/2$ we may define the set $L \subset \mathscr{O}$ in Theorem 7.2 as
$$L := \{(R, Z) \,|\, R\underline{b}/2 \le Z \le 2\overline{b}R,\ \underline{r}/2 \le R \le 2\overline{r}\}.$$
Clearly, this set is compact in $\mathscr{O}$. The couple $(f(\beta)\mathfrak{r}, g(\beta)\mathfrak{z})$ created from the strong solution $(\beta, \mathfrak{r}, \mathfrak{z}, \mathbf{U})$ is ranging, according to Theorem 4.1, in the compact set
$$K = \{(Z, R) | \underline{b}R \le Z \le \overline{b}R,\ \underline{r} \le R \le \overline{r}\}$$
which is included in the interior of the set $L$.

2. Theorem 7.2 is valid also with the slip (Navier) boundary conditions for velocity, or if $\Omega$ is a periodic cell (with periodic boundary conditions), cf. Remark 3.1.

3. Function $p$ with $\gamma_\pm > 1$ introduced in formula (1.14) provides an example of a pressure stemming from partial pressure constitutive laws $P_\pm$, cf. (1.13), which satisfies together with its Helmoholtz function $H$ given in (3.12) all assumptions of Theorem 7.2.

Before starting the proof of Theorem 7.2, we shall need an auxiliary lemma dealing with the estimates of differences of concentrations.

**Lemma 7.3.** *Let* $(\alpha, \mathbf{u})$, $(\beta, \mathbf{U})$,
$$\alpha, \beta \in L^\infty(Q_T) \cap C(\overline{I}; L^1(\Omega)),\ \mathbf{u}, \mathbf{U} \in L^2(I; W^{1,2}_0(\Omega))$$
*be solutions of the pure transport equation in* $\mathscr{D}'(Q_T)$*, cf. (1.7). Suppose moreover that*
$$\nabla\beta \in L^\infty(Q_T),\ \mathrm{div}\mathbf{U} \in L^1(I; L^\infty(\Omega)).$$

*Then for any* $\delta > 0$ *and for all* $\tau \in \overline{I}$,
$$\int_\Omega (\alpha - \beta)^2(\tau, \cdot)\ \mathrm{d}x - \int_\Omega (\alpha - \beta)^2(0, \cdot)\ \mathrm{d}x \le \delta \int_0^\tau \|\mathbf{u} - \mathbf{U}\|^2_{W^{1,2}(\Omega)} \mathrm{d}t + \frac{c}{\delta}\int_0^\tau a(t) \int_\Omega (\alpha - \beta)^2\ \mathrm{d}x\mathrm{d}t, \quad (7.4)$$
*where* $a = \|\alpha\|^2_{L^\infty(\Omega)} + \|\beta\|^2_{L^\infty(\Omega)} + \|\nabla\beta\|^2_{L^\infty(\Omega)} + \|\mathrm{div}\mathbf{U}\|_{L^\infty(\Omega)} \in L^1(0, T)$ *and* $c > 0$ *is a universal constant.*

**Proof of Lemma 7.3.** To prove the lemma we shall use the nowadays classical DiPerna-Lions regularizing technique, cf. [9].

According to the assumptions, the couples $(\alpha, \mathbf{u})$ and $(\beta, \mathbf{U})$ satisfy

$$\partial_t \alpha + \mathbf{u} \cdot \nabla \alpha = 0 \text{ in } \mathscr{D}'(Q_T),$$

$$\partial_t \beta + \mathbf{u} \cdot \nabla \beta = (\mathbf{u} - \mathbf{U}) \cdot \nabla \beta \text{ a.e. in } Q_T.$$

Since $\mathbf{u}$ admits a zero trace on $\partial\Omega$, the first equation holds in $\mathscr{D}'(I \times R^3)$ provided we extend $\mathbf{u}$ and $\alpha$ by 0 outside $\Omega$. We may therefore regularize the first equation by using the standard mollifiers over the space variable in order to get, in particular,

$$\partial_t [\alpha]_\varepsilon + \mathbf{u} \cdot \nabla [\alpha]_\varepsilon = r_\varepsilon(\varrho, \mathbf{u}) \ \text{ a.e. in } Q_T,$$

where $[\alpha]_\varepsilon$ denotes the mollified $\alpha$ and $r_\varepsilon(\varrho, \mathbf{u}) = \mathbf{u} \cdot \nabla [\alpha]_\varepsilon - [\mathbf{u} \cdot \nabla \alpha]_\varepsilon$. Combining the third and the second equation, we deduce

$$\partial_t([\alpha]_\varepsilon - \beta) + \mathbf{U} \cdot \nabla([\alpha]_\varepsilon - \beta) = r_\varepsilon + f_\varepsilon \ \text{ a.e.in } Q_T,$$

where $f_\varepsilon = (\mathbf{U} - \mathbf{u}) \cdot \nabla [\alpha]_\varepsilon$. Multiplying the latter identity by $[\alpha]_\varepsilon - \beta$ and integrating over $(0, \tau) \times \Omega$, we get

$$\frac{1}{2} \int_\Omega ([\alpha]_\varepsilon - \beta)^2(\tau) \ \mathrm{d}x - \frac{1}{2} \int_\Omega ([\alpha]_\varepsilon - \beta)^2(0) \ \mathrm{d}x \tag{7.5}$$
$$-\frac{1}{2} \int_0^\tau \int_\Omega \mathrm{div}\mathbf{U}([\alpha]_\varepsilon - \beta)^2 \ \mathrm{d}x\mathrm{d}t = \int_0^\tau \int_\Omega (r_\varepsilon + f_\varepsilon)([\alpha]_\varepsilon - \beta) \ \mathrm{d}x\mathrm{d}t$$

for all $\tau \in \overline{I}$.

Due to the chain of identities,

$$\int_\Omega f_\varepsilon([\alpha]_\varepsilon - \beta) \ \mathrm{d}x = \frac{1}{2} \int_\Omega (\mathbf{U} - \mathbf{u}) \cdot \nabla([\alpha_\varepsilon] - \beta)^2 \ \mathrm{d}x + \int_\Omega (\mathbf{U} - \mathbf{u}) \cdot \nabla \beta([\alpha_\varepsilon] - \beta) \ \mathrm{d}x$$

$$= -\frac{1}{2} \int_\Omega \mathrm{div}(\mathbf{U} - \mathbf{u})([\alpha_\varepsilon] - \beta)^2 \ \mathrm{d}x + \int_\Omega (\mathbf{U} - \mathbf{u}) \cdot \nabla \beta([\alpha_\varepsilon] - \beta) \ \mathrm{d}x,$$

we have

$$\int_\Omega f_\varepsilon([\alpha_\varepsilon] - \beta) \ \mathrm{d}x \leq \frac{\delta}{2} \|\mathrm{div}(\mathbf{U} - \mathbf{u})\|^2_{L^2(\Omega)} + \frac{\delta}{2} \|\mathbf{U} - \mathbf{u}\|^2_{L^2(\Omega)}$$
$$+\frac{1}{\delta}(\|[\alpha]_\varepsilon - \beta\|^2_{L^\infty(\Omega)} + \|\nabla\beta\|^2_{L^2(\Omega)}) \int_\Omega ([\alpha_\varepsilon] - \beta)^2 \ \mathrm{d}x,$$

with any $\delta > 0$, where we have used the Hölder and Young inequalities, and where $\|[\alpha]_\varepsilon\|_{L^\infty(\Omega)} \leq \|\alpha\|_{L^\infty(\Omega)}$. Moreover, by virtue of the Friedrichs lemma about commutators, cf. [9],

$$r_\varepsilon \to 0 \text{ in } L^2(Q_T) \text{ as } \varepsilon \to 0+;$$

whence

$$\int_0^\tau \int_\Omega r_\varepsilon([\alpha_\varepsilon] - \beta) \ \mathrm{d}x \to 0 \text{ as } \varepsilon \to 0+.$$

Consequently, the identity (7.5) yields the inequality

$$\frac{1}{2}\int_\Omega ([\alpha]_\varepsilon - \beta)^2(\tau) \ \mathrm{d}x - \frac{1}{2}\int_\Omega ([\alpha]_\varepsilon - \beta)^2(0) \ \mathrm{d}x \le \frac{\delta}{2}\int_0^\tau \|\mathrm{div}(\mathbf{U} - \mathbf{u})\|^2_{L^2(\Omega)}\mathrm{d}t$$

$$+\frac{\delta}{2}\int_0^\tau \|\mathbf{U} - \mathbf{u}\|^2_{L^2(\Omega)}\mathrm{d}t + \int_0^\tau \Big[\Big(\frac{1}{\delta}(\|[\alpha]_\varepsilon - \beta\|^2_{L^\infty(\Omega)} + \|\nabla\beta\|^2_{L^2(\Omega)}) + \|\mathrm{div}\mathbf{U}\|_{\mathrm{L}^\infty(\Omega)}\Big)\int_\Omega ([\alpha]_\varepsilon - \beta)^2 \ \mathrm{dx}\Big]\mathrm{dt}.$$

We get the inequality (7.4) after the limit passage $\varepsilon \to 0$ in the latter relation.

The proof of Lemma 7.3 is complete.

**Proof of Theorem 7.2.**

We define the essential and residual sets corresponding to $(\alpha, \varrho, z)$ with respect to compact set $L$ as follows

$$\Omega_{\mathrm{ess}}(t) := \{x \in \Omega \,|\, (f(\alpha)\varrho(t), g(\alpha)z(t)) \in L\}, \ \Omega_{\mathrm{res}}(t) := \Omega \setminus \Omega_{\mathrm{ess}}(t). \tag{7.6}$$

Similarly we decompose an integrable function $h$ to its essential $[h]_{\mathrm{ess}}$ and residual $[h]_{\mathrm{res}}$ parts by setting

$$[h]_{\mathrm{ess}}(t,x) = h(t,x)1_{\Omega_{\mathrm{ess}}(t)}(x), \ [h]_{\mathrm{res}}(t,x) = h(t,x)1_{\Omega_{\mathrm{res}}(t)}(x). \tag{7.7}$$

We are now in position to estimate the right hand side of the relative energy inequality (6.1). We shall do it in four steps.

**Step 1:** *The material derivative term (the first term).*
First, we split the first term in the remainder (6.1) as follows:

$$\int_0^\tau \int_\Omega (\mathbf{U} - \mathbf{u}) \cdot \Big[(\varrho + z - \mathfrak{r} - \mathfrak{z})\partial_t \mathbf{U} + \Big((\varrho + z)\mathbf{u} - (\mathfrak{r} + \mathfrak{z})\mathbf{U}\Big) \cdot \nabla\mathbf{U}\Big] dx\mathrm{d}t \tag{7.8}$$

$$= \int_\Omega (\mathbf{U} - \mathbf{u}) \cdot \Big[(\varrho + z - \mathfrak{r} - \mathfrak{z})\partial_t \mathbf{U} + \Big((\varrho + z) - (\mathfrak{r} + \mathfrak{z})\Big)\mathbf{U} \cdot \nabla\mathbf{U}\Big] dx\mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega (\varrho + z)(\mathbf{u} - \mathbf{U}) \cdot \nabla\mathbf{U} \cdot (\mathbf{U} - \mathbf{u}) dx\mathrm{d}t,$$

where

$$\int_0^\tau \int_\Omega (\varrho + z)(\mathbf{u} - \mathbf{U}) \cdot \nabla\mathbf{U} \cdot (\mathbf{U} - \mathbf{u}) dx\mathrm{d}t \tag{7.9}$$

$$\le \int_0^\tau a(t)\mathscr{E}\Big(\alpha, \varrho, z, \mathbf{u}\Big|\beta, \mathfrak{r}, \mathfrak{z}, \mathbf{U}\Big)\mathrm{d}t, \text{ where } a = 2\|\nabla\mathbf{U}\|_{L^\infty(\Omega;R^9)} \in L^2(0,T).$$

In order to handle the first term on the right hand side of (7.8), we write

$$\varrho - \mathfrak{r} = \frac{1}{f(\alpha)}\big(f(\alpha)\varrho - f(\beta)\mathfrak{r}\big) + \big(\frac{1}{f(\alpha)} - \frac{1}{f(\beta)}\big)\mathfrak{r}$$

meaning that

$$|\varrho - \mathfrak{r}| \leq c\Big(\big|f(\alpha)\varrho - f(\beta)\mathfrak{r}\big| + |\alpha - \beta|\Big), \tag{7.10}$$

where $c > 0$ depends on $\underline{\alpha}, \overline{\alpha}, \overline{\mathfrak{r}}$, $\max_{\zeta\in[\underline{\alpha},\overline{\alpha}]}\Big(\Big|\frac{f'(\zeta)}{f^2(\zeta)}\Big|\Big)$. Similarly,

$$|z - \mathfrak{z}| \leq c\Big(\big|g(\alpha)z - g(\beta)\mathfrak{z}\big| + |\alpha - \beta|\Big) \tag{7.11}$$

where $c > 0$ depends on $\underline{\alpha}, \overline{\alpha}, \overline{\mathfrak{z}}$, $\max_{\zeta\in[\underline{\alpha},\overline{\alpha}]}\Big(\Big|\frac{g'(\zeta)}{g^2(\zeta)}\Big|\Big)$.

We shall treat separately the "essential part" and the "residual part". We estimate the essential part as follows

$$\int_0^\tau \int_\Omega [1]_{\rm ess}\Big((\varrho - \mathfrak{r}) + (z - \mathfrak{z})\Big)\Big(\partial_t \mathbf{U} + \mathbf{U}\cdot\nabla\mathbf{U}\Big)\cdot(\mathbf{U} - \mathbf{u}) \ {\rm d}x{\rm d}t \tag{7.12}$$

$$\leq c\int_0^\tau \|\partial_t\mathbf{U} + \mathbf{U}\cdot\nabla\mathbf{U}\|_{L^\infty(\Omega;R^3)}\Big\|\mathbf{u} - \mathbf{U}\Big\|_{L^2(\Omega;R^3)}$$

$$\times\Big(\Big\|f(\alpha)\varrho - f(\beta)\mathfrak{r}\|_{L^2(\Omega_{\rm ess}(t))} + \|g(\alpha)z - g(\beta)\mathfrak{z}\|_{L^2(\Omega_{\rm ess}(t))} + \|\alpha - \beta\|_{L^2(\Omega)}\Big){\rm d}t$$

$$\leq \delta\int_0^\tau \Big\|\mathbf{u} - \mathbf{U}\Big\|^2_{L^2(\Omega;R^3)}{\rm d}t + \frac{c}{\delta}\int_0^\tau a(t)\mathscr{E}\Big(\alpha, \varrho, z, \mathbf{u}\Big|\beta, \mathfrak{r}, \mathfrak{z}, \mathbf{U}\Big){\rm d}t,$$

where

$$a = \|\partial_t\mathbf{U} + \mathbf{U}\cdot\nabla\mathbf{U}\|^2_{L^\infty(\Omega;R^3)} \in L^1(0,T).$$

Concerning the residual part, we may write

$$\int_0^\tau \int_\Omega [1]_{\rm res}\Big((\varrho - \mathfrak{r}) + (z - \mathfrak{z})\Big)\Big(\partial_t \mathbf{U} + \mathbf{U}\cdot\nabla\mathbf{U}\Big)\cdot(\mathbf{U} - \mathbf{u}) \ {\rm d}x{\rm d}t$$

$$\int_0^\tau \int_\Omega [1]_{\rm res}\Big[(\varrho - \mathfrak{r})1_{\{\varrho\leq D\}}(\varrho) + (\varrho - \mathfrak{r})1_{\{\varrho> D\}}(\varrho) + (z - \mathfrak{z})1_{\{\varrho\leq D\}}(z) + (z - \mathfrak{z})1_{\{r> D\}}(z)\Big]$$

$$\times\Big(\partial_t \mathbf{U} + \mathbf{U}\cdot\nabla\mathbf{U}\Big)\cdot(\mathbf{U} - \mathbf{u}){\rm d}x{\rm d}t$$

with any $D > 0$ which we take sufficiently large (larger than $\overline{\mathfrak{r}}+\overline{\mathfrak{z}}$). The typical terms of this development will be estimated as follows:

$$\int_0^\tau \int_\Omega [1]_{\rm res}1_{\{\varrho\leq D\}}(\varrho)(\varrho - \mathfrak{r})(\partial_t\mathbf{U} + \mathbf{U}\cdot\nabla\mathbf{U})\cdot(\mathbf{U} - \mathbf{u}) \ {\rm d}x{\rm d}t$$

$$\leq 2D\int_0^\tau\int_\Omega [1]_{\rm res}\Big|\partial_t\mathbf{U}+\mathbf{U}\cdot\nabla\mathbf{U}\Big|\ \Big|\mathbf{U}-\mathbf{u}\Big|\ \,{\rm d}x{\rm d}t$$

$$\leq 2D\int_0^\tau \|\partial_t\mathbf{U}+\mathbf{U}\cdot\nabla\mathbf{U}\|_{L^\infty(\Omega;R^3)}\Big\|[1]_{\rm res}\Big\|_{L^2(\Omega)}\Big\|\mathbf{u}-\mathbf{U}\Big\|_{L^2(\Omega;R^3)}{\rm d}t$$

$$\leq \delta\int_0^\tau \Big\|\mathbf{u}-\mathbf{U}\Big\|^2_{L^2(\Omega;R^3)}{\rm d}t+\frac{c}{\delta}\int_0^\tau a(t)\mathscr{E}\Big(\alpha,\varrho,z,\mathbf{u}\Big|\beta,\mathfrak{r},\mathfrak{z},\mathbf{U}\Big){\rm d}t,$$

where $a$ is given in (7.12).

Finally,

$$\int_0^\tau\int_\Omega 1_{\{\varrho>D\}}(\varrho)[1]_{\rm res}(\varrho-\mathfrak{r})(\partial_t\mathbf{U}+\mathbf{U}\cdot\nabla\mathbf{U})\cdot(\mathbf{U}-\mathbf{u})\ \,{\rm d}x{\rm d}t$$

$$\leq 2\int_0^\tau\int_\Omega 1_{\{\varrho>D\}}(\varrho)[1]_{\rm res}\sqrt{\varrho}\Big|\partial_t\mathbf{U}+\mathbf{U}\cdot\nabla\mathbf{U}\Big|\sqrt{\varrho}\Big|\mathbf{U}-\mathbf{u}\Big|\ \,{\rm d}x{\rm d}t$$

$$\leq \int_0^\tau \|\partial_t\mathbf{U}+\mathbf{U}\cdot\nabla\mathbf{U}\|_{L^\infty(\Omega;R^3)}\ \|[\varrho]_{\rm res}\|^{1/2}_{L^1(\Omega)}\ \|\varrho\Big(\mathbf{u}-\mathbf{U}\Big)^2\|^{1/2}_{L^1(\Omega)}{\rm d}t$$

$$\leq c\int_0^\tau a(t)\mathscr{E}\Big(\alpha,\varrho,z,\mathbf{u}\Big|\beta,\mathfrak{r},\mathfrak{z},\mathbf{U}\Big){\rm d}t$$

with the same $a$ as before. In all above three formulas, we have employed Lemma 7.1 in the passage to their last lines.

Resuming the first step, the material derivative term obeys the bound

$$\int_0^\tau\int_\Omega(\mathbf{U}-\mathbf{u})\cdot\Big[(\varrho+z-\mathfrak{r}-\mathfrak{z})\partial_t\mathbf{U}+\Big((\varrho+z)\mathbf{u}-(\mathfrak{r}+\mathfrak{z})\mathbf{U}\Big)\cdot\nabla\mathbf{U}\Big]dx{\rm d}t \tag{7.13}$$

$$\leq \delta\int_0^\tau\|\mathbf{u}-\mathbf{U}\|^2_{W^{1,2}(\Omega;R^3)}{\rm d}t+\frac{c}{\delta}\int_0^\tau a(t)\mathscr{E}\Big(\alpha,\varrho,z,\mathbf{u}\Big|\beta,\mathfrak{r},\mathfrak{z},\mathbf{U}\Big){\rm d}t,$$

where $\delta>0$ is an arbitrary number, $c=c(f,g,\underline{\alpha},\overline{\alpha},\underline{\mathfrak{r}},\ \underline{\mathfrak{z}},\overline{\mathfrak{r}},\overline{\mathfrak{z}})>0$, and

$$a=\|\nabla\mathbf{U}\|_{L^\infty(\Omega;R^9)}+\|\partial_t\mathbf{U}+\mathbf{U}\cdot\nabla\mathbf{U}\|^2_{L^\infty(\Omega;R^3)}\in L^1(0,T).$$

**Step 2:** *The pressure/divergence* $\mathbf{U}$ *term (the second term)*

Similarly as in the first step, we split the second term in the remainder (6.1) to the esssential and residual parts. The essential part is bounded by

$$\int_0^\tau\int_\Omega[1]_{\rm ess}\Big(p(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})-p(f(\alpha)\varrho,g(\alpha)z)-\partial_R p(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})(f(\beta)\mathfrak{r}-f(\alpha)\varrho)$$

$$-\partial_Z p(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})(g(\beta)\mathfrak{z}-g(\alpha)z)\Big){\rm d}x{\rm d}t$$

$$\leq c\int_0^\tau\int_\Omega[1]_{\rm ess}\Big(\Big|f(\alpha)\varrho-f(\beta)\mathfrak{r}\Big|^2+\Big|g(\alpha)z-f(\beta)\mathfrak{z}\Big|^2\Big)\ \,{\rm d}x{\rm d}t\leq c\int_0^\tau a(t)\mathscr{E}(\alpha,\varrho,z,\mathbf{u}|\beta,\mathfrak{r},\mathfrak{z},\mathbf{U}){\rm d}t$$

by virtue of the second order Taylor formula, where we have used Lemma 7.1 to get the last inequality.

Employing Lemma 7.1 (and the Young inequality), we deduce the pointwise bound,

$$[1]_{\rm res}\Big|p(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})-p(f(\alpha)\varrho,g(\alpha)z)-\partial_R p(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})(f(\beta)\mathfrak{r}-f(\alpha)\varrho)$$

$$-\partial_Z p(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})(g(\beta)\mathfrak{z}-g(\alpha)z)\Big|\le c[1]_{\rm res}\mathscr{E}(\alpha,\varrho,z,\mathbf{u}|\beta,\mathfrak{r},\mathfrak{z},\mathbf{U})$$

in order to estimate the residual part,

$$\int_0^\tau\int_\Omega[1]_{\rm res}\Big(p(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})-p(f(\alpha)\varrho,g(\alpha)z)-\partial_R p(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})(f(\beta)\mathfrak{r}-f(\alpha)\varrho)$$

$$-\partial_Z p(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})(g(\beta)\mathfrak{z}-g(\alpha)z)\Big){\rm d}x{\rm d}t$$

$$\le c\int_0^\tau a(t)\mathscr{E}(\alpha,\varrho,z,\mathbf{u}|\beta,\mathfrak{r},\mathfrak{z},\mathbf{U}){\rm d}t.$$

In the above two formulas, $c$ is a positive number dependent of $\overline{R}$, $\underline{\mathfrak{r}},\overline{\mathfrak{r}},\underline{\mathfrak{z}},\overline{\mathfrak{z}}$ and $|\partial_R p|_{C^1(L)}+|\partial_Z p|_{C^1(L)}$ while

$$a(t)=1.$$

Resuming estimates in Step 2,

$$\int_0^\tau\int_\Omega\Big(p(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})-p(f(\alpha)\varrho,g(\alpha)z)-\partial_R p(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})(f(\beta)\mathfrak{r}-f(\alpha)\varrho)$$

$$-\partial_Z p(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})(g(\beta)\mathfrak{z}-g(\alpha)z)\Big){\rm d}x{\rm d}t$$

$$\le c\int_0^\tau a(t)\mathscr{E}(\alpha,\varrho,z,\mathbf{u}|\beta,\mathfrak{r},\mathfrak{z},\mathbf{U}){\rm d}t \tag{7.14}$$

**Step 3:** *The terms containing the Helmholtz function (the third and fourth terms)*

By the same token as in Steps 1 and 2,

$$\int_0^\tau\int_\Omega\Big(f(\alpha)\varrho-f(\beta)\mathfrak{r}\Big)(\mathbf{U}-\mathbf{u})\cdot\nabla\partial_R H(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})dx{\rm d}t$$

$$+\int_\Omega\Big(g(\alpha)z-g(\beta)\mathfrak{z}\Big)(\mathbf{U}-\mathbf{u})\cdot\nabla\partial_Z H(f(\beta)\mathfrak{r},g(\beta)\mathfrak{z})dx$$

$$\le\delta\int_0^\tau\|\mathbf{u}-\mathbf{U}\|^2_{L^2(\Omega)}+\frac{c}{\delta}\int_0^\tau a(t)\mathscr{E}(\alpha,\varrho,z,\mathbf{u}|\beta,\mathfrak{r},\mathfrak{z},\mathbf{U}){\rm d}t, \tag{7.15}$$

with any $\delta>0$, where

$$a(t)=1$$

and $c > 0$ depends on $\underline{\alpha}$, $\overline{\alpha}$, $\underline{\mathfrak{r}}, \overline{\mathfrak{r}}, \underline{\mathfrak{z}}, \overline{\mathfrak{z}}$ and $|\partial_R H|_{C^1(L)} + |\partial_Z H|_{C^1(L)}$.

**Step 4:** *Conclusion*

Summarizing Steps 1-3, we get the following bound for the remainder (6.1),

$$\int_0^\tau \mathscr{R}_{\alpha,\beta}(\varrho, z, \mathbf{u}|\mathfrak{r}, \mathfrak{z}, \mathbf{U})\mathrm{d}t \leq \delta \int_0^\tau \left\|\mathbf{u} - \mathbf{U}\right\|^2_{W^{1,2}(\Omega;R^3)} \mathrm{d}t + \frac{c}{\delta} \int_0^\tau a(t)\mathscr{E}\Big(\alpha, \varrho, z, \mathbf{u}\Big|\beta, \mathfrak{r}, \mathfrak{z}, \mathbf{U}\Big)\mathrm{d}t$$

with any $\delta > 0$ and $c > 0$ dependent on $\underline{\alpha}$, $\overline{\alpha}$, $\underline{\mathfrak{r}}, \overline{\mathfrak{r}}, \underline{\mathfrak{z}}, \overline{\mathfrak{z}}$, $|\partial_R H|_{C^1(L)} + |\partial_Z H|_{C^1(L)}$, $|\partial_R p|_{C^1(L)} + |\partial_Z p|_{C^1(L)}$, $\max_{\zeta \in [\underline{\alpha}, \overline{\alpha}]}\Big(\Big|\frac{f'(\zeta)}{f^2(\zeta)}\Big|\Big)$, $\max_{\zeta \in [\underline{\alpha}, \overline{\alpha}]}\Big(\Big|\frac{g'(\zeta)}{g^2(\zeta)}\Big|\Big)$,

$$a = \|\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{U}\|^2_{L^\infty(\Omega;R^3)} + \|\nabla \mathbf{U}\|_{L^\infty(\Omega;R^9)} + 1 \in L^1(0,T).$$

Coming back with this estimate to relative energy inequality (2.8) with the remainder given by (6.1) and adding to it the inequality (7.4) we end up with the relative energy inequality presented in Theorem 7.2. This completes the proof.

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