Observability in Target Motion Analysis from the Sum or the Difference of Ranges with Two Stationary Sensors

Annie-Claude Perez, Claude Jauffret

Observability in Target Motion Analysis from the Sum or the Difference of Ranges with Two Stationary Sensors

Annie-Claude Pérez,
Université de Toulon, Aix Marseille Univ, CNRS, IM2NP,
Toulon, France
CS 60584, 83041 TOULON Cedex 9, France
annie-claude.perez@univ-tln.fr

Claude Jauffret,
Université de Toulon, Aix Marseille Univ, CNRS, IM2NP,
Toulon, France
CS 60584, 83041 TOULON Cedex 9, France
jauffret@univ-tln.fr

***Abstract*— We address in this paper the problem of observability in target motion analyses (TMA), when the measurements are a sum or a difference of ranges between a target and two motionless observers. Necessary and sufficient conditions of observability are given in time difference of arrival (TDOA) situation and in a bistatic situation.**

Keywords—Target motion analysis, tracking, range-only, TDOA, bistatic, observability, Fisher information

I. INTRODUCTION

In any problem of estimation, the first step consists of analyzing observability of the parameter of interest. When the noise-free measurements depend linearly on the parameter (one deal with a linear system), the task is easy: the parameter is observable when the matrix linking the measurement vector to the parameter vector is full-ranked. In a large class of probability laws of additive noise, this is equivalent to the regularity of the Fisher information matrix (FIM) [1]. When the system is nonlinear, the definition of observability must be extended: the parameter can be either locally observable, if it is observable in an open vicinity, or globally observable if this open vicinity is the whole space in which it is defined. Analyzing observability requires more mathematical tools.

Systems encountered in target motion analysis (TMA) problems are nonlinear in most cases. The observability is hence a hard task unless the noise-free system can be transformed into an equivalent linear system, as in bearings-only TMA [2, 3, 4, 5]. In the TMA literature, two main ways are used for this purpose: the rank of the FIM can be studied as in [6] [7], or we can use analytical tools as in [8, 9, 10]. When the FIM rank serves as observability criterion, some precaution must be taken: the rank must be constant in an open subspace containing the parameter [11]. If not, the conclusion can be wrong. For example, in range-only TMA, the FIM can be singular at an isolated point and still this point is observable. Conversely, the FIM can be full ranked, and the parameter remains unobservable [1].

We claim that using the determinant of the FIM (or its rank) to conclude observability can mislead.

In this paper, we intend analyzing observability of the trajectory of a target in constant velocity (CV) motion, in two TMA problems: when the measurement are time differences of arrival (TDAO) and when the measurements are those collected in a bistatic configuration (as claimed in [12]). Our main tool comes from the polynomial algebra. The analysis will be made in continuous time.

In section II, the notations are given.

Section III is devoted to the case of range difference measurements. In this part, we give conclusions about observability in TDOA-TMA.

The bistatic configuration is addressed in section IV.

A conclusion follows.

II. NOTATIONS AND FUNDAMENTAL LEMMAS

A. Notations

A target (T) and two observers (O1) and (O2) are in the same plane, given a Cartesian system. The target on in CV motion all along the scenario, while the observers are stationary. The scenario starts at time $t=0$ and finishes at time $t=T_f$.

The observers are located at $O_1=\begin{bmatrix}-a & 0\end{bmatrix}^{\mathsf{T}}$ and $O_2=\begin{bmatrix}a & 0\end{bmatrix}^{\mathsf{T}}$ respectively.

The position of the target is $P_T(t)=[x_T(t)\ y_T(t)]^{\mathsf{T}}$ at time t. Its velocity is $V_T=\dfrac{dP_T(t)}{dt}=[\dot{x}_T\ \ \dot{y}_T]^{\mathsf{T}}$. The ranges relatively to the two observers are $r_1(t)=\|O_1P_T(t)\|$ and $r_2(t)=\|O_2P_T(t)\|$, respectively. Figure 1 illustrates the used notations.

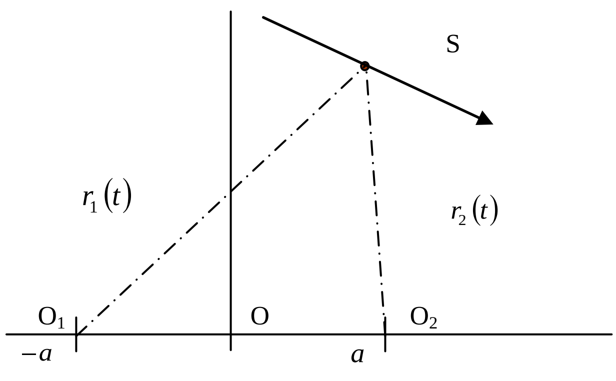

Fig. 1: typical scenario

B. Lemmas

Lemma 1:

$r_1(t)r_2(t)$ is a constant function or a polynomial

function of degree 2 if, and only if $\left[r_1(t)+\varepsilon r_2(t)\right]^2$ *with* $\varepsilon=\pm 1$ *is a polynomial function.*

Proof:

$$\left[r_1(t)+\varepsilon r_2(t)\right]^2 = r_1(t)^2 + r_2(t)^2 + 2\varepsilon\, r_1(t) r_2(t)$$
$$= \left(x_T(t)+a\right)^2 + \left(x_T(t)-a\right)^2 + 2y_T^2(t)$$
$$+2\varepsilon\sqrt{\left[\left(x_T(t)+a\right)^2+y_T^2(t)\right]\left[\left(x_T(t)-a\right)^2+y_T^2(t)\right]}$$

If $\left[r_1(t)+\varepsilon r_2(t)\right]^2$ is a polynomial function, then $\left[r_1(t)+\varepsilon r_2(t)\right]^2 - r_1(t)^2 - r_2(t)^2 = 2\varepsilon\, r_1(t) r_2(t)$ is a polynomial function too. Let us compute its degree:

Since $r_1^2(t) r_2^2(t) = \left[\left(x_T(t)+a\right)^2+y_T^2(t)\right]\left[\left(x_T(t)-a\right)^2+y_T^2(t)\right]$, the highest degree is 4. The coefficient of t^4 is $\ddot{x}_T^2+\ddot{y}_T^2$. If $\ddot{x}_T^2+\ddot{y}_T^2 \neq 0$, then $r_1^2(t) r_2^2(t)$ is a polynomial function of degree 4 ; therefore the degree of $r_1(t) r_2(t)$ is 2. Otherwise, $\ddot{x}_T = \ddot{y}_T = 0$, $r_1^2(t) r_2^2(t)$ is a constant function, and $r_1(t) r_2(t)$ too.

The converse is obvious.

QED

In the coming study, we will distinguish three cases:

$r_1(t) r_2(t)$ is a constant

$r_1(t) r_2(t)$ is a polynomial function.

$r_1(t) r_2(t)$ is neither a constant, nor a polynomial function.

Lemma 2:

$r_1(t) r_2(t)$ *is a constant function if, and only if the target is stationary.*

Proof

Suppose that $r_1(t) r_2(t)$ hence $\left[r_1(t) r_2(t)\right]^2$ is a constant function. Since

$\left[r_1(t) r_2(t)\right]^2 = \left[\left(x_T(t)+a\right)^2+y_T^2(t)\right]\left[\left(x_T(t)-a\right)^2+y_T^2(t)\right]$, all the coefficients of the non-null powers of t are null, in particular the coefficient of t^4, which is $\ddot{x}_T^2+\ddot{y}_T^2$. Hence the target is fixed.

The converse is obvious.

QED

Remark 1: In the proofs of the following lemmas and propositions, we will use the following property:

If the target is moving in the (Ox) axis, then

$r_1(t) = \left|x_T(t)+a\right|$ and $r_2(t) = \left|x_T(t)-a\right|$. Three situations can occur:

- The target is between the two sensors: $r_1(t) = x_T(t)+a$ and $r_2(t) = -x_S(t)+a$.
  Consequently, $r_1(t)-r_2(t) = 2x_T(t)$, and $r_1(t)+r_2(t) = 2a$
- The target is moving in $\left]a,\infty\right[$, then $r_1(t) = x_T(t)+a$ and $r_2(t) = x_T(t)-a$. We deduce that $r_1(t)-r_2(t) = 2a$, and $r_1(t)+r_2(t) = 2x_T(t)$.
- The target is in $\left]-\infty,-a\right[$, and we get
  $r_1(t)-r_2(t) = -2a$, and $r_1(t)+r_2(t) = -2x_T(t)$.

Lemma 3:

$r_1(t) r_2(t)$ *is a polynomial function of degree 2 if, and only if the target is moving in the line* (O_1, O_2), *or in the perpendicular bisector of the segment* $[O_1, O_2]$.

Proof

If $r_1(t) r_2(t)$ is a polynomial function of degree 2, then, three numbers α, β, γ exist such that $r_1(t) r_2(t) = \gamma(t-\alpha)(t-\beta)$. The numbers α, β can be real or not, whereas γ is a real number.

a) If α and β are two real numbers, then

$$\left[\left(x_T(t)+a\right)^2+y_T^2(t)\right]\left[\left(x_T(t)-a\right)^2+y_T^2(t)\right] = \gamma^2(t-\alpha)^2(t-\beta)^2$$

Consequently,
$\left[\left(x_T(\alpha)+a\right)^2+y_T^2(\alpha)\right]\left[\left(x_T(\alpha)-a\right)^2+y_T^2(\alpha)\right] = 0$.

Hence $y_T^2(\alpha)=0$; for similar reasons $y_T^2(\beta)=0$. We end up with $y_T(\alpha) = y_T(\beta) = 0$.

Let us examine two subcases :

If $\alpha=\beta$, then
$\left[\left(x_T(t)+a\right)^2+y_T^2(t)\right]\left[\left(x_T(t)-a\right)^2+y_T^2(t)\right] = \gamma^2(t-\alpha)^4$.

As a consequence $\begin{cases}\left(x_T(t)+a\right)^2+y_T^2(t) = \gamma_1(t-\alpha)^2\\ \left(x_T(t)-a\right)^2+y_T^2(t) = \gamma_2(t-\alpha)^2\end{cases}$, with $\gamma_1\gamma_2 = \gamma^2$.

We deduce that $x_T(\alpha) = a$ and $x_T(\alpha) = -a$, which is incompatible.

Hence $\alpha \neq \beta$. The polynomial function $y_T(t)$ - of degree 0 or 1 - is null for two different values. It is necessarily the null polynomial function. The target is hence moving in the (Ox) axis.

b) α and β are not two real numbers. Since $r_1(t) r_2(t)$ is real, $\beta = \alpha^*$, and $r_1(t) r_2(t) = \gamma(t-\alpha)(t-\alpha^*)$. We get

$$\left[\left(x_T(t)+a\right)^2+y_T^2(t)\right]\left[\left(x_T(t)-a\right)^2+y_T^2(t)\right] = \gamma^2(t-\alpha)^2(t-\alpha^*)^2.$$

Necessarily, two real numbers γ_1 exist γ_2 such that

$$\begin{cases} \left(x_T(t)+a \right)^2 + y_T^2(t) = \gamma_1 (t-\alpha)(t-\alpha^*) \\ \left(x_T(t)-a \right)^2 + y_T^2(t) = \gamma_2 (t-\alpha)(t-\alpha^*) \\ \gamma_1 \gamma_2 = \gamma \end{cases}$$

We have $\left(x_T(\alpha)+a \right)^2 + y_T^2(\alpha) = 0$,

and $\left(x_T(\alpha)-a \right)^2 + y_T^2(\alpha) = 0$.

We deduce that $\left(x_T(\alpha)+a \right)^2 = \left(x_T(\alpha)-a \right)^2$, hence that $4x_T(\alpha)a = 0$. But $x_T(t)$ is a polynomial function of degree 0 or 1. We conclude that $x_T(t)=0$, $\forall$ t. The target is moving in the (Oy) axis.

Conversely, if the target is moving in the line (O_1, O_2), then

$r_1(t)r_2(t) = \sqrt{\left(x_T(t)+a \right)^2 \left(x_T(t)-a \right)^2} = \varepsilon \left(x_T(t)+a \right)\left(x_T(t)-a \right)$, with $\varepsilon = \pm 1$.

If the target is moving in the perpendicular bisector of the segment $[O_1, O_2]$, then

$r_1(t)r_2(t) = \sqrt{\left[a^2 + y_T^2(t) \right]^2} = \left[a^2 + y_T^2(t) \right]$.

QED

III. RANGE DIFFERENCE

A. Analysis

In this section, the available measurement is the difference of ranges between each observer and the target. The observability analysis will be conducted with $\left[r_1(t) - r_2(t) \right]^2$ and its sign.

Proposition 1:

The range difference is null if, and only if the target is in the perpendicular bisector of the segment $[O_1, O_2]$. In this case, the trajectory of the target is not observable.

Proof

Suppose that the difference of range is null.

$r_1(t) = r_2(t) \Leftrightarrow r_1(t)^2 = r_2(t)^2$, that is, $\left(x_T(t)+a \right)^2 + y_T^2(t) = \left(x_T(t)-a \right)^2 + y_T^2(t)$.

We have hence $x_T(t)a = 0$, that is $x_T(t) = 0$. The target is in the (0y) axis.

Conversely, if the target is in the (0y) axis, the ranges are equal.

QED

Proposition 2:

The range difference is a non-null constant if, and only if the target is motionless out of the perpendicular bisector of the segment $[O_1, O_2]$, or the target is in the line spanned by the two sensors, out of the segment $[O_1, O_2]$. In both cases, the trajectory of the target is not observable. Nevertheless, we are able to know if the target is on the left or on the right of the segment $[O_1, O_2]$.

Proof

If the difference of range is a non-null constant, its square is a non-null constant, as well:

$r_1(t)^2 + r_2(t)^2 - 2r_1(t)r_2(t) = \text{cste}$. Since $r_1(t)^2 + r_2(t)^2$ is a polynomial function, $r_1(t)r_2(t)$ is a constant or a polynomial of degree 2.

From Lemma 2, if $r_1(t)r_2(t)$ is a constant, then the target is stationary.

From Lemma 3, if $r_1(t)r_2(t)$ is a polynomial of degree 2, then the target is moving in the axis (Ox) or in the axis (Oy). The last case must be discarded since the range difference is non-null. Hence the target is moving in the axis (Ox). Remark 1 helps us to conclude.

The converse is obvious.

QED

Proposition 3:

The range difference is a polynomial function of degree 1 if, and only if the target is moving between the two sensors. In this case, the trajectory of the target is observable: their initial abscissa is its half of the first difference of range, and its velocity is half of the time coefficient.

Proof

If the range difference is a polynomial function of degree 1, then necessarily $r_1(t)r_2(t)$ is a polynomial of degree 2. Hence the target is moving in the line (O_1, O_2), or in the perpendicular bisector of the segment $[O_1, O_2]$ (see Lemma 3). Since the range difference is non-null, the target is in the line (O_1, O_2). From Remark 1, the target is moving between the two sensors.
The converse is obvious.

QED

Proposition 4:

The square of the range difference is neither a polynomial function nor a constant function if, and only if the target is in CV motion out of the line $(O_1\ O_2)$ and the perpendicular bisector of the segment $[O_1, O_2]$. In this case, its trajectory is observable up to the axial symmetry around the line $(O_1\ O_2)$ and up to the perpendicular bisector of the segment $[O_1, O_2]$.

Proof:

Let a ghost-target G detected by the same measurements:

$$r_{G1}(t) - r_{G2}(t) = r_1(t) - r_2(t), \ \forall t \in [0, T_f]$$

$$\Rightarrow \ \left[r_{G1}(t) - r_{G2}(t) \right]^2 = \left[r_1(t) - r_2(t) \right]^2, \ \forall t \in [0, T_f]$$

$$\Leftrightarrow \quad r_1^2(t)+r_2^2(t)-2r_1(t)r_2(t)=r_{G1}^2(t)+r_{G2}^2(t)-2r_{G1}(t)r_{G2}(t)$$

$$\Leftrightarrow \quad r_1^2(t)+r_2^2(t)-r_{G1}^2(t)-r_{G2}^2(t)=2r_1(t)r_2(t)-2r_{G1}(t)r_{G2}(t) \quad (1)$$

Hence, $r_1(t)r_2(t)-r_{G1}(t)r_{G2}(t)$ is a polynomial function of degree 2.

But, by assumption, $r_1(t)r_2(t)$ is not a polynomial function; hence $r_{G1}(t)r_{G2}(t)$ as well.

Consequently, it exists two polynomial functions of degree 2, say $P_2(t)$ and $Q_2(t)$, and a non-polynomial function $g(t)$ such that

$$r_1(t)r_2(t)=P_2(t)+g(t), \quad (2)$$

$$r_{G1}(t)r_{G2}(t)=Q_2(t)+g(t). \quad (3)$$

Since $r_1^2(t)r_2^2(t)$ and $r_{G1}^2(t)r_{G2}^2(t)$ are polynomial functions of degree 4, the functions $2P_2(t)g(t)+g^2(t)$ and $2Q_2(t)g(t)+g^2(t)$ are polynomial functions of degree 4.

We deduce that

$$r_1^2(t)r_2^2(t)-r_{G1}^2(t)r_{G2}^2(t)=P_2^2(t)-Q_2^2(t)+2[P_2(t)-Q_2(t)]g(t)$$
.

At this point, we have to discuss about the nature of $P_2(t)-Q_2(t)$:

1) If $P_2(t)-Q_2(t)\neq 0$
$g(t)=\dfrac{r_1^2(t)r_2^2(t)-r_{G1}^2(t)r_{G2}^2(t)-P_2^2(t)+Q_2^2(t)}{2[P_2(t)-Q_2(t)]}$ is a rational fraction (the ratio of two polynomial functions), say

$$g(t)=\frac{R_4(t)}{S_4(t)} \text{ (irreducible fraction).} \quad (4)$$

Now, eq (1) implies that

$$\left[r_1^2(t)+r_2^2(t)-r_{G1}^2(t)-r_{G2}^2(t)\right]^2=4\left[r_1(t)r_2(t)-r_{G1}(t)r_{G2}(t)\right]^2$$

$$\Leftrightarrow \quad 8r_1(t)r_2(t)r_{G1}(t)r_{G2}(t)=4\left[r_1^2(t)r_2^2(t)+r_{G1}^2(t)r_{G2}^2(t)\right]^2 -\left[r_1^2(t)+r_2^2(t)-r_{G1}^2(t)-r_{G2}^2(t)\right]^2$$

which means that $r_1(t)r_2(t)r_{G1}(t)r_{G2}(t)$ is a polynomial function of degree 4.

From (2) and (3),

$$r_1(t)r_2(t)r_{G1}(t)r_{G2}(t)=P_2(t)Q_2(t)+g(t)[P_2(t)+Q_2(t)+g(t)]$$

$$\Leftrightarrow r_1(t)r_2(t)r_{G1}(t)r_{G2}(t)-P_2(t)Q_2(t)=\frac{R_4(t)}{S_4(t)}[P_2(t)+Q_2(t)+g(t)]$$

$$\Leftrightarrow R_4(t)g(t)=S_4(t)\left[r_1(t)r_2(t)r_{G1}(t)r_{G2}(t)-P_2(t)Q_2(t)\right] -R_4(t)[P_2(t)+Q_2(t)].$$

Hence, $R_4(t)g(t)$ is a polynomial function, which is equal to $\dfrac{R_4^2(t)}{S_4(t)}$ (see (4)). Consequently, $S_4(t)$ divides $R_4^2(t)$, which means that $S_4(t)$ and $R_4(t)$ have common roots. This is not the assumption of (4). This case must be discarded.

2) Hence $P_2(t)-Q_2(t)=0$
From (1), we get readily

$$r_1^2(t)+r_2^2(t)=r_{G1}^2(t)-r_{G2}^2(t) \quad (5)$$

$$2r_1(t)r_2(t)=2r_{G1}(t)r_{G2}(t) \quad (6)$$

Let us develop the left term of (5), then its the right terms :

$$r_1^2(t)+r_2^2(t)=2a^2+2x_T^2(t)+2y_T^2(t),$$

$$r_{G1}^2(t)+r_{G2}^2(t)=2a^2+2x_G^2(t)+2y_G^2(t).$$

Consequently, (5) is equivalent to

$$x_T^2(t)+y_T^2(t)=x_G^2(t)+y_G^2(t), \quad \forall t\in[0, T_f] \quad (7)$$

which means that the two targets are in the same circle whose center is the middle point of $[O_1\ O_2]$.

Now, we exploit (6), which yields

$$\left(x_T^2(t)+y_T^2(t)\right)^2+2a^2\left(y_T^2(t)-x_T^2(t)\right)=\left(x_G^2(t)+y_G^2(t)\right)^2+2a^2\left(y_G^2(t)-x_G^2(t)\right)$$

Using (7), we get

$$y_T^2(t)-x_T^2(t)=y_G^2(t)-x_G^2(t) \quad (8)$$

Eq. (7) and (8) allows us to conclude that

$x_T(t)=\pm x_G(t)$, and $y_G(t)=\pm y_S(t)$.

QED.

The knowledge of the sign of the range difference removes the symmetry relatively to the perpendicular bisector of the segment $[O_1, O_2]$, and can state the following property:

Proposition 5:

The range difference is neither a constant, nor a polynomial function of degree 1 if, and only if the target is out of the line $(O_1\ O_2)$ and out of the perpendicular bisector of the segment $[O_1, O_2]$.
In this case, its trajectory is observable up to the axial symmetry around the line $(O_1\ O_2)$.

Remark 2: with noisy measurements, a statistical test allows us to decide if the difference of range is always null, constant and non-null, linear or non-linear. Depending on the response of the test, we can know, before tenting to estimate the trajectory, if it is observable or not.

B. Application: TDOA [6]

In a TDOA, the measurement is the difference of ranges up to a multiplicative constant that is the speed of the wave (in the medium of interest). So, the previous analysis is directly applicable. Proposition 3 and Proposition 5 give us necessary and sufficient conditions of observability. Moreover, if a third observer is added to the scenario, the trajectory of any target in CV motion is observable from TDOA.

IV. RANGE SUM

A. Analysis

Here also, the observability analysis will be based upon the square of $r_1(t)+r_2(t)$:

$$[r_1(t)+r_2(t)]^2 = r_1(t)^2 + r_2(t)^2 + 2r_1(t)r_2(t).$$

Note that the sum is always positive.

We can straightforwardly give the following result:

Proposition 6:

The square of the range sum is a polynomial function of degree 2 if, and only if the target is either in the line $(O_1\ O_2)$ but out of the segment $[O_1, O_2]$, or in the perpendicular bisector of the segment $[O_1, O_2]$.

Proof

From Lemma 1 and Lemma 3, if the square of the range sum is a polynomial function of degree 2, then the target is either in the line $(O_1\ O_2)$, or in the perpendicular bisector of the segment $[O_1, O_2]$.

a) If the target is in the perpendicular bisector of the segment $[O_1, O_2]$, then $x_T(t)=0, \forall t$ and $y_T(t)\neq 0$. Consequently, $r_1(t)+r_2(t)=2\sqrt{a^2+y_T^2(t)}$, and $[r_1(t)+r_2(t)]^2=4[a^2+y_T^2(t)]$.

The identification of the coefficients of the polynomial function $[r_1(t)+r_2(t)]^2$ allows us to recover the initial position and the velocity of the target.

b) If the target is in the line $(O_1\ O_2)$, Remark 1 allows us to finish the proof :

- If the target is moving in $[O_1, O_2]$, then $\forall t\ r_1(t)+r_2(t)=2a$. This case must be discarded.
- If the target is moving in the line $(O_1\ O_2)$, but out of $[O_1, O_2]$, then $r_1(t)+r_2(t)=\pm 2x_T(t)\ ,\forall t$.

Conversely, if the target is either in the line $(O_1\ O_2)$ but out of the segment $[O_1, O_2]$, or in the perpendicular bisector of the segment $[O_1, O_2]$, the sum is a polynomial function of degree 2.

QED

During this proof, we got three results:

Proposition 7:

The square of the range sum is a polynomial function of degree 2 and the range sum is not a polynomial function of degree 1 if, and only if the target is in the perpendicular bisector of the segment $[O_1, O_2]$.
In this case, the trajectory of the target is observable up to the symmetry around the center of $[O_1, O_2]$.

The proof is the part a) of the proof of Proposition 6.

Proposition 8:

The range sum is a polynomial function of degree 1 if, and only if the target is in the line $(O_1\ O_2)$ but out of the segment $[O_1, O_2]$. In this case, the trajectory of the target is observable up to the symmetry around the center of $[O_1, O_2]$.

The proof is the part b) of the proof of Proposition 6 (second indent).

Proposition 9:

a) *The square of the range sum is neither a constant function nor a polynomial function if, and only if the target is in CV motion, out of the line $(O_1\ O_2)$ and out of the perpendicular bisector of the segment $[O_1, O_2]$.*
b) *The target is observable up to the axial symmetries around these two lines.*

Proof

The part a) is a consequence of Proposition 7 and Proposition 8. Only, the part b) has to be proved.

Let a ghost-target G detected by the same measurements:

$$r_{G1}(t)+r_{G2}(t)=r_1(t)+r_2(t),\ \forall t\in[0, T_f]$$

$$\Leftrightarrow\quad [r_{G1}(t)+r_{G2}(t)]^2=[r_1(t)+r_2(t)]^2,\ \forall t\in[0, T_f]$$

$$\Leftrightarrow\quad r_1^2(t)+r_2^2(t)+2r_1(t)r_2(t)=r_{G1}^2(t)+r_{G2}^2(t)+2r_{G1}(t)r_{G2}(t)$$
$$\Leftrightarrow\quad r_1^2(t)+r_2^2(t)-r_{G1}^2(t)-r_{G2}^2(t)=-2r_1(t)r_2(t)+2r_{G1}(t)r_{G2}(t)$$

Since this last equation is eq. (1) up to the sign of the right term, the rest of the proof is the same as the proof of Proposition 4: We end up with the two following equalities:

$$\begin{cases} x_T^2(t)+y_T^2(t)=x_G^2(t)+y_G^2(t) \\ x_T^2(t)-y_T^2(t)=x_G^2(t)-y_G^2(t) \end{cases}$$

As a consequence, $x_G(t)=\pm x_T(t)$, and $y_G(t)=\pm y_T(t)$.

Conversely, if $x_G(t)=\pm x_T(t)$, and $y_G(t)=\pm y_T(t)$, then

$$r_{G1}(t)+r_{G2}(t)=r_1(t)+r_2(t),\ \forall t\in[0, T_f].$$

QED.

Proposition 10:

The range sum is a constant function if, and only if the target is either in the segment $[O_1, O_2]$, or stationary. In this case, the trajectory of the target is not observable.

B. Application: the passive bistatic radar [12]

In a bistatic configuration, O1 is the transmitter, and O2 is the receiver.
The transmitter sends a signal continuously and the target plays the role of mirror: it reflects the signal emitted by the transmitter toward the receiver. If the signal $s_E(t)$ is a single

tone, that is $s_E(t)=a\sin(2\pi f_0 t+\varphi)$, the signal reflected by the target is the received signal $s_R(t) = a'\sin(2\pi f_0[t-\tau_1-\tau_2+\varphi])$ with $\tau_i = \frac{r_i}{c}$ with i=1,2. It follows that the instantaneous frequency at time t is

$$f(t)= f_0\left[1-\frac{\dot r_1(t)+\dot r_2(t)}{c}\right] \text{ for } t\in\left[0,T_f\right].$$

The emitted frequency f_0 and the wave speed c are assumed to be known.

We have $\dot r_1(t)+\dot r_2(t)=\frac{P_1(t)}{r_1(t)}+\frac{P_2(t)}{r_2(t)}$, with

$P_1(t)=(x_T(t)+a)\dot x_T+y_T(t)\dot y_T$, and

$P_2(t)=(x_T(t)-a)\dot x_T+y_T(t)\dot y$, that are two polynomial functions of degree 1.

If a ghost-target G exists, far from the transmitter and receiver $r_{G1}(t)$ and $r_{G2}(t)$ respectively, then

$$f(t)= f_0\left[1-\frac{\dot r_{G1}(t)+\dot r_{G2}(t)}{c}\right].$$

Note that

$$\frac{P_1(t)}{r_1(t)}+\frac{P_2(t)}{r_2(t)}=\frac{P_{G1}(t)}{r_{G1}(t)}+\frac{P_{G2}(t)}{r_{G2}(t)} \Rightarrow \left[\frac{P_1(t)}{r_1(t)}+\frac{P_2(t)}{r_2(t)}\right]^2=\left[\frac{P_{G1}(t)}{r_{G1}(t)}+\frac{P_{G2}(t)}{r_{G2}(t)}\right]^2$$

$$\Leftrightarrow \underbrace{\left(\frac{P_1(t)}{r_1(t)}\right)^2+\left(\frac{P_2(t)}{r_2(t)}\right)^2}_{\text{rational fraction}}+2\frac{P_1(t)P_2(t)}{r_1(t)r_2(t)}=\underbrace{\left(\frac{P_{G1}(t)}{r_{G1}(t)}\right)^2+\left(\frac{P_{G2}(t)}{r_{G2}(t)}\right)^2}_{\text{rational fraction}}+2\frac{P_{G1}(t)P_{G2}(t)}{r_{G1}(t)r_{G2}(t)}.$$

A part of the observability study will be conducted with the products $r_1(t)r_2(t)$ and $r_{G1}(t)r_{G2}(t)$.

Proposition 11

$f(t)$ is a constant function, if and only if the target is stationary or is moving along the line (O_1, O_2). The trajectory of the target is unobservable.

Proof

If $f(t)$ is a constant function, then $r_1(t)+r_2(t)=\alpha t+\beta$. Hence, $[r_1(t)+r_2(t)]^2=\alpha^2t^2+2\alpha\beta t+\beta^2$. Therefore, $r_1(t)r_2(t)$ is a constant function or a polynomial function of degree 2.

If $r_1(t)r_2(t)$ is a constant function, then the target is stationary (see Lemma 2). We have no way to know its location.

If $r_1(t)r_2(t)$ is a polynomial function of degree 2, then T is moving in the line (O_1, O_2), or in the perpendicular bisector of the segment $[O_1, O_2]$(see Lemma 3). If T is moving in the perpendicular bisector of the segment $[O_1, O_2]$, then $x_T(t)=0$, and consequently $r_1(t)=r_2(t)=\sqrt{a^2+y_T^2(t)}$ which is not a polynomial function of degree 1. This case must be rejected. If T is moving in the line (O_1, O_2).

The converse is obvious.

QED

Proposition 12

$f(t)$ is not a constant function, if and only if the target is moving out of the line (O_1, O_2). In this case, and only in this case, the trajectory of the target is observable up to the axial symmetries around the line $(O_1\, O_2)$ and the perpendicular bisector of the segment $[O_1, O_2]$.

Proof

The first statement is the converse of Proposition 11. We only have to prove that the trajectory of the target is observable.

Two cases must be distinguished:

1) The target is moving along the perpendicular bisector of the segment $[O_1, O_2]$. Lemma 3 tells us that $r_1(t)r_2(t)$ is a polynomial function of degree 2. If a ghost-target G exists, $r_{G1}(t)r_{G2}(t)$ is necessarily a polynomial function of degree 2. Therefore, G is moving in the perpendicular bisector of the segment $[O_1, O_2]$, $r_{G1}(t)=r_{G2}(t)=\sqrt{a^2+y_G^2(t)}$.
Since $\dot r_{G1}(t)+\dot r_{G2}(t)=\dot r_1(t)+\dot r_2(t)$, we have

$$\frac{y_G(t)\dot y_G}{\sqrt{a^2+y_G^2(t)}}=\frac{y_T(t)\dot y_T}{\sqrt{a^2+y_T^2(t)}} \qquad (9)$$

$$\Leftrightarrow \frac{y_T(t)\dot y_T}{y_G(t)\dot y_G}=\sqrt{\frac{a^2+y_T^2(t)}{a^2+y_G^2(t)}}.$$

$$\Rightarrow \left[\frac{y_T(t)\dot y_T}{y_G(t)\dot y_G}\right]^2=\frac{a^2+y_T^2(t)}{a^2+y_G^2(t)}$$

$$\Leftrightarrow y_T^2(t)\dot y_T^2\left[a^2+y_G^2(t)\right]=y_G^2(t)\dot y_G^2\left[a^2+y_T^2(t)\right]$$

$$\Leftrightarrow a^2\left[y_T^2(t)\dot y_T^2-y_G^2(t)\dot y_G^2\right]=y_T^2(t)y_G^2(t)\left[\dot y_G^2-\dot y_T^2\right]$$

$$\Leftrightarrow y_T^2(t)y_G^2(t)\left(\dot y_T^2-\dot y_G^2\right)+a^2\left[y_T^2(t)\dot y_T^2-y_G^2(t)\dot y_G^2\right]=0 \qquad (10)$$

This polynomial function of degree 4 is equal to zero. Then all its coefficients are null, in particular the one of t^4, that is $\dot y_T\dot y_G\left(\dot y_T^2-\dot y_G^2\right)=0$. Because $\dot y_T\neq 0$ and $\dot y_G\neq 0$, we have $\dot y_G^2=\dot y_T^2$. Introducing this equality in (14), we get $y_G^2(t)=y_T^2(t)$.

Eq (9) implies that $y_T(t)\dot y_T$ et $y_G(t)\dot y_G$ have the same sign. Hence $y_G(t)=\varepsilon\, y_T(t)$ with $\varepsilon=\pm1$.

2) The target is moving out of the perpendicular bisector of the segment $[O_1, O_2]$.

The ghost-target is necessarily moving off the perpendicular bisector of the segment $[O_1, O_2]$ too (other $r_{G1}(t)r_{G2}(t)$ would be a polynomial function of degree 2, and $r_1(t)r_2(t)$ as well).

We have :

$$\underbrace{\left(\frac{P_1}{r_1}\right)^2+\left(\frac{P_2}{r_2}\right)^2}_{\text{rational fraction}}+2\frac{P_1 P_2}{r_1 r_2}=\underbrace{\left(\frac{P_{G1}}{r_{G1}}\right)^2+\left(\frac{P_{G2}}{r_{G2}}\right)^2}_{\text{rational fraction}}+2\frac{P_{G1} P_{G2}}{r_{G1} r_{G2}}$$

$$\Leftrightarrow \quad \underbrace{\left(\frac{P_1}{r_1}\right)^2+\left(\frac{P_2}{r_2}\right)^2-\left(\frac{P_{G1}}{r_{G1}}\right)^2-\left(\frac{P_{G2}}{r_{G2}}\right)^2}_{=F}=2\frac{P_{G1} P_{G2}}{r_{G1} r_{G2}}-2\frac{P_1 P_2}{r_1 r_2}$$

$$\Rightarrow F^2=\underbrace{4\left(\frac{P_{G1} P_{G2}}{r_{G1} r_{G2}}\right)^2+4\left(\frac{P_1 P_2}{r_1 r_2}\right)^2}_{\text{rational fraction}}-4\frac{P_1 P_2 P_{G1} P_{G2}}{r_1 r_2 r_{G1} r_{G2}}$$

Consequently, $r_1 r_2 r_{G1} r_{G2}$ is a polynomial function.

Since neither $r_1 r_2$ nor $r_{G1} r_{G2}$ are polynomial functions, a positive real number β exists such that $r_{G1} r_{G2}=\beta r_1 r_2$. Hence, $r_{G1}{}^2 r_{G2}{}^2=\beta^2 r_1{}^2 r_2{}^2$.

Consequently, either $\begin{cases} r_{G1}{}^2=\delta r_1{}^2 \\ r_{G2}{}^2=\gamma r_2{}^2 \end{cases}$ or $\begin{cases} r_{G1}{}^2=\delta r_2{}^2 \\ r_{G2}{}^2=\gamma r_1{}^2 \end{cases}$, with $\delta>0,\ \gamma>0$ et $\delta\gamma=\beta^2$.

Let us study each case:

$$\begin{cases} r_{G1}{}^2=\delta r_1{}^2 \\ r_{G2}{}^2=\gamma r_2{}^2 \end{cases} \Leftrightarrow \begin{cases} (x_G(t)+a)^2+y_G^2(t)=\delta\left[(x_T(t)+a)^2+y_T^2(t)\right] \\ (x_G(t)-a)^2+y_G^2(t)=\gamma\left[(x_T(t)-a)^2+y_T^2(t)\right] \end{cases}$$

which implies, by difference, that

$$4x_G(t)a=\delta\left[(x_T(t)+a)^2+y_T^2(t)\right]-\gamma\left[(x_T(t)-a)^2+y_T^2(t)\right]$$

$$\Leftrightarrow \delta\left[(x_T(t)+a)^2+y_T^2(t)\right]-\gamma\left[(x_T(t)-a)^2+y_T^2(t)\right]-4x_G(t)a=0$$

All the coefficients of this polynomial function are null, in particular the one of t^2 :

$(\delta-\gamma)(\dot{x}_T^2+\dot{y}_T^2)=0$, that yields $\delta=\gamma=\beta$. Hence, $\begin{cases} r_{G1}(t)=\beta r_1(t) \\ r_{G2}(t)=\beta r_2(t) \end{cases}$.

The equality $\dot{r}_1(t)+\dot{r}_2(t)=\dot{r}_{G1}(t)+\dot{r}_{G2}(t)$ implies that $\beta=1$:

$r_{G1}(t)=r_1(t)$ and $r_{G2}(t)=r_2(t)$.

If $\begin{cases} r_{G1}{}^2=\delta r_2{}^2 \\ r_{G2}{}^2=\gamma r_1{}^2 \end{cases}$, similar computations end up with $r_{G1}(t)=r_2(t)$ and $r_{G2}(t)=r_1(t)$.

These two cases yield $r_{G1}(t)+r_{G2}(t)=r_1(t)+r_2(t)$. Proposition 9 completes the proof.

QED

Remark 3: In [7], the rank of the FIM was used as observability criterion. The conclusion of the authors of [7] is that the trajectory is observable if the target does not travel toward the transmitter or toward the receiver. But, the authors did not care if the rank of the FIM was constant in an open vicinity of the parameter of interest. It is not the case. Proposition 12 contradicts their conclusion.

V. CONCLUSION

In this paper, we addressed the question about observability of the trajectory of a target in CV motion, when the sum of range (between the target and two stationary observers) or their difference is available.

Through an analysis based on the algebraic nature of the noise-free measurement, we end up with the following non-ambiguous answers:

1) A motionless target is unobservable from the knowlegde of the sum or the difference of ranges.

2) A moving target is observable under the following conditions:

- In TDOA (two passive sensors), the trajectory of the target is observable if, and only if the target does not travel in the perpendicular bisector of the segment $[O_1, O_2]$ or in the line $(O_1\ O_2)$ minus the segment $[O_1, O_2]$. This is up to the axial symmetry around the line $(O_1\ O_2)$.
- In a bistatic situation (a transmitter and a receiver), the observability is guaranteed if and only if the target is moving out of the line (O_1, O_2). This is up to the axial symmetries around the line $(O_1\ O_2)$ and the perpendicular bisector of the segment $[O_1, O_2]$.

REFERENCES

[1] Jauffret, C., "Observability and Fisher Information Matrix in Nonlinear Regression," IEEE Transactions on Aerospace and Electronic Systems, 43, 2, pp. 756–759, Apr. 2007.

[2] Jauffret, C., and Pillon, D., "Observability in Passive Target Motion Analysis," IEEE Transactions on Aerospace and Electronic Systems, 32, 4, pp. 1290–1300, Oct. 1996.

[3] Fogel, E., and Gavish, M., "Nth-order Dynamics Target Observability from Angle Measurements," IEEE Transactions on Aerospace and Electronic Systems, 24, 3, pp. 305–308, May 1988.

[4] Nardone, S.C., and Aidala, V.J. "Observability Criteria for Bearings-Only Target Motion Analysis," IEEE Transactions on Aerospace and Electronic Systems, 17, 2, pp. 162–166, Mar. 1981.

[5] Le Cadre, J.P. and Jauffret, C., "Discrete-Time Observability and Estimability Analysis for Bearings-Only Target Motion Analysis," IEEE Transactions on Aerospace and Electronic Systems, 33, 1, pp. 178–201, Jan. 1997.

[6] J.F. Arnold, Y. Bar-Shalom, R. Estrada and R. Mucci, "Target Parameter Estimation Using Measurements Acquired with a Small Number of Sensors", IEEE Journal of Oceanic Engineering, Vol. 8, No. 3, pp. 163–171, Jul. 1983.

[7] Y.C. Xiao, P. Wei and T. Yuan, "Observability and Performance Analysis of Bi/Multi-Static Doppler-Only Radar", IEEE Transactions on Aerospace and Electronic Systems, Vol. 46, No. 4, pp. 1654-1667, Oct. 2010.

[8] Pillon, D., Pérez-Pignol, A.C., and Jauffret, C. "Observability: Range-Only vs. Bearings-Only Target Motion Analysis for a Leg by Leg Observer's Trajectory.," IEEE Transactions on Aerospace and Electronic Systems, Vol 52, No 4, pp 1667-1678, Aug 2016.

[9] Pignol A.C., Jauffret C. et Pillon D., "Properties of range-only target

motion analysis," The 16th International Conference on Information Fusion, Istanbul, Turkey, Jul 2013.

[10] Jauffret C., Pérez A.C., Pillon D., "Observability: Range-Only Versus Bearings-Only Target Motion Analysis When the Observer maneuvers Smoothly," IEEE Transactions on Aerospace and Electronic Systems, Vol 53, No 6, pp 2814-2832, Dec 2017.

[11] Rothenberg T.J., "Identification in Parametric Models," Econometrica, Vol. 39, No. 3, pp 577-595, May 1971.

[12] Jauffret C., Pérez A.C., Blanc-Benon P., Tanguy H., "Doppler-Only Target Motion Analysis in a High Duty Cycle Sonar System," 19th International Conference on Information Fusion, Heidelberg, Germany, Jul 2016.