Range-Only Target Motion Analysis: Observability when the Observer Maneuvers Smoothly

Annie-Claude Pérez, Claude Jauffret
Aix Marseille Univ, Univ Toulon, CNRS, IM2NP,
Toulon, France
jauffret, annie-claude.perez@univ-tln.fr

Denis Pillon
retired
Thonon-les-Bains, France
pillon_denis@orange.fr

***Abstract*—this paper is the companion-paper of another paper presented in FUSION'17 concerning bearings-only target motion analysis (BOTMA). In this one, bearing data are replaced by range data: we study observability in range-only target motion analysis (ROTMA). When the observer is in constant turn motion, the target's trajectory is observable, as in BOTMA. If the observer is in constant acceleration motion, necessary and sufficient conditions of observability in ROTMA are proven to be not the same than in BOTMA, although the equality of the ranks of the respective Fisher information matrices. As in BOTMA, the rendezvous routes of the observer and the target play a crucial role. When the system is not observable, ghost-targets (the ones at the same range than the true target) are identified.**

***Index Terms*— Target motion analysis, tracking, range-only, observability, constant turn motion, constant acceleration motion, sonar, radar, electronic support measurement.**

I. INTRODUCTION

THIS PAPER addresses the observability analysis in range-only target motion analysis (ROTMA) when the target is assumed to be in constant velocity (CV) motion whereas the observer maneuvers smoothly. It completes another one [5] in which we analyze observability in bearing-only target motion analysis in the same context. These both papers are the continuation of [2] where the trajectory of the observer is simpler. Here, we will prove that when the observer is in constant turn (CT) motion, the target is observable in ROTMA. This maneuver was previously used in ROTMA [3], [4]. When the observer is in constant acceleration (CA) motion, several situations can occur: the target can be observable or not. This proves the existence of ineffective maneuvers in ROTMA, as in BOTMA [1]. We propose a criterion of observability, based upon range measurements. When the trajectory of the target is not observable, the set of ghost-targets is given.

The paper is organized as follows:
In Section II, the problem formulation and notations are presented.
In section III we recall the observer kinematic models and give condition on range of rendezvous route when the observer is in CA motion.
In section IV, we prove observability of the target when the observer is in CT motion.
Section V is devoted to observers in CA motion. For this type of scenario, the trajectory of the target is not necessarily observable. Therefore we give necessary and sufficient observability conditions. In unobservable cases, we identify the set of ghost-targets. Illustrative examples are given.
The conclusion follows.

II. PROBLEM FORMULATION AND NOTATIONS

A. Definitions and notations

In a plane given Cartesian system, a target (T) moves in CV motion and an observer (O) maneuvers smoothly. The scenario starts at time $t=0$ and finishes at time $t=T_f$. At time t the position $P_O(t)=[x_O(t)\ y_O(t)]^\intercal$ and velocity $V_O(t)=\dfrac{dP_O(t)}{dt}=[\dot{x}_O(t)\ \dot{y}_O(t)]^\intercal$ of the observer are concatenated into the vector $X_O(t)=\left[x_O(t) \quad y_O(t) \quad \dot{x}_O(t) \quad \dot{y}_O(t)\right]^\intercal$.

For the target, at time t the position $P_T(t)=[x_T(t)\ y_T(t)]^\intercal$, and velocity $V_T=\dfrac{dP_T(t)}{dt}=[\dot{x}_T\ \dot{y}_T]^\intercal$ verify $P_T(t)=P_T(0)+tV_T$ and are concatened into $X_T(t)=\left[x_T(t) \quad y_T(t) \quad \dot{x}_T \quad \dot{y}_T\right]^\intercal$.

The trajectory of the target relative to the observer is given by its relative position $P_{OT}(t)=P_T(t)-P_O(t)=[x_{OT}(t)\ y_{OT}(t)]^\intercal$ and by its relative velocity $V_{OT}(t)=\dfrac{dP_{OT}(t)}{dt}=[\dot{x}_{OT}(t)\ \dot{y}_{OT}(t)]^\intercal$. We define the vector
$X_{OT}(t)=X_T(t)-X_O(t)=\left[x_{OT}(t) \quad y_{OT}(t) \quad \dot{x}_{OT}(t) \quad \dot{y}_{OT}(t)\right]^\intercal$.
The vector $X_{OT}(0)$ that entirely defines the target's trajectory will be simply denoted as X_{OT} subsequently.

The relative position of the target can be expressed in polar

coordinate $P_{OT}(t)=r(t)\begin{bmatrix}\sin\theta(t)\\ \cos\theta(t)\end{bmatrix}$, where $r(t)$ and $\theta(t)$ are the range and the bearing at t, respectively. As well the initial relative velocity of the target is $V_{OT}(0)=v_r\begin{bmatrix}\sin h_r\\ \cos h_r\end{bmatrix}$. The polar coordinates of $V_{OT}(0)$ are then (v_r, h_r).

B. What is observability ?

The previous notation are extended in order to emphasize this dependence: $r(t)$ will be denoted $r(t, X_{OT})$. We recall that the target's trajectory is declared observable in ROTMA if the following statement is true: $\forall t\in[0, T_f]$, $r(t, X)=r(t, X_{OT}) \Rightarrow X=X_{OT}$. Otherwise, the trajectory is said to be unobservable: at least one vector $X_{OG}=[x_{OG}(0) \quad y_{OG}(0) \quad \dot{x}_{OG} \quad \dot{y}_{OG}]^{\mathrm{T}}$ (defining a CV motion) different from X_{OT} exists such that $r(t, X_{OG})=r(t, X_{OT})$.

The vector $X_G=X_{OG}+X_O$ defines the "virtual" trajectory of a "ghost-target", denoted G. Each "ghost-target" is defined by $P_G(0)=[x_G(0) \quad y_G(0)]^{\mathrm{T}}$, $V_G=[\dot{x}_G \ \dot{y}_G]^{\mathrm{T}}$, $P_G(t)=P_G(0)+tV_G=[x_G(t) \ y_G(t)]^{\mathrm{T}}$, and $X_G(t)=[x_G(t) \quad y_G(t) \quad \dot{x}_G \quad \dot{y}_G]^{\mathrm{T}}$, with the convention $X_G=X_G(0)$.

Observability analysis has two aims:

a) give a necessary and sufficient condition to have unicity of X_{OT} (the trajectory of the target is observable) and the way to know that from measurements,
b) when the trajectory is unobservable, characterize the set of the ghost-targets, that is X_{OG}.

This will conduct our paper.

Due to the length of the paper, some proofs given in [5] and [6] are omitted here.

III. Observer cinematic models

Let us start our study by defining the two types of maneuvers considered here: the constant turn motion, that is the observer travels in an arc of a circle at constant speed, and the constant acceleration motion. We detail hereafter the equations of these two motions.

A. CT motion

The observer turns around a fixed point $P_C=[x_C \quad y_C]^{\mathrm{T}}$ at range $\rho>0$, with a constant turn rate $\omega\neq 0$ (positive if the motion of the observer is clockwise) and an "initial phase" φ relative to North, at $t=0$. Its speed is constant. As a consequence, at time t, the location of the observer is given by $P_O(t)=P_C+\rho\begin{bmatrix}\sin(\omega t+\varphi)\\ \cos(\omega t+\varphi)\end{bmatrix}$. Without loss of generality, we will assume that $x_C=y_C=0$.

B. CA motion

At any time *t*, the position of the observer is $P_O(t)=P_O(0)+tV_O(0)+\frac{t^2}{2}\Gamma$, where $V_O(0)$ is the initial velocity and $\Gamma=[\gamma_x \quad \gamma_y]^{\mathrm{T}}$ is the (non-zero) acceleration vector. The relative position of the target with respect to the observer is

$$P_{OT}(t)=P_{OT}(0)+tV_{OT}(0)-\frac{t^2}{2}\Gamma \quad (1)$$

Without loss of generality (and in order to simplify the coming computations), we will assume that $\gamma_x<0$ and $\gamma_y=0$. Indeed, a suitable rotation of the entire scenario allows us to be in this case.

When the target is in CA motion, analyzing observability conducts us to consider special scenarios: the rendezvous routes whose definition is now given

The target and the observer are said to be on a rendezvous route (RDVR), when they are collocated at a time t_c.

From now, we will assume that t_c is not in $[0, T_f]$.

Proposition 1: General properties of RDVR

If O (in CA motion) and T are on an RDVR, then

- *either $P_{OT}(0)$, $V_{OT}(0)$ and Γ are collinear,*
- *or $P_{OT}(0)$ and Γ are noncollinear, and $V_{OT}(0)$ and Γ are noncollinear.*

See [5], for the proof.

Definition: The two types of RDVR

The RDVRs are called rendezvous routes of type I (RDVR-I), when $P_{OT}(0)$, $V_{OT}(0)$, and Γ are collinear. When $P_{OT}(0)$ and Γ are noncollinear, and $V_{OT}(0)$ and Γ are noncollinear as well, the RDVRs are called rendezvous routes of type II (RDVR-II).

Note that for the RDVR-II, $P_{OT}(0)$ *and* $V_{OT}(0)$ can be collinear.

The following three propositions give sufficient and necessary conditions of the two types of RDVD.

Proposition 2: Condition of RDVR-I

Assume that $P_{OT}(0)$, $V_{OT}(0)$ and Γ are collinear; that is, $P_{OT}(0)=\eta\Gamma$ (with $\eta\neq 0$) and $V_{OT}(0)=\lambda\Gamma$. O and T are on an RDVR if and only if $\lambda^2\geq -2\eta$.

See [5], for the proof.

Proposition 3: Condition of RDVR-II.

Assume that $P_{OT}(0)$ and Γ are noncollinear, and $V_{OT}(0)$ and Γ are noncollinear as well.

O and T are on an RDVR if and only if $\gamma_x=2\frac{\dot{y}_{OT}(0)}{y_{OT}^2(0)}\left[x_{OT}(0)\dot{y}_{OT}(0)-\dot{x}_{OT}(0)y_{OT}(0)\right]$.

See [5], for the proof.

Proposition 4: Criterion on range of RDVR-I

The target and the observer are on RDVR-I if and only if a

scalar $\dot{r}$ *exist such that, for* $t \in [0, T_f]$, either $r(t) - \frac{1}{2}t^2\gamma_x = r_0 + t\dot{r}$ *or* $r(t) + \frac{1}{2}t^2\gamma_x = r_0 + t\dot{r}$ *and* $\dot{r}^2 \geq -2r_0\gamma_x$.

The proof is given in [6].

IV. Observability when observer is in CT motion

The following result is established when the noise-free range are continuously available during $t \in [0, T_f]$

Proposition 5: Observability when observer is in CT motion
If the observer is traveling along an arc of a circle, then any target moving with a constant velocity is observable by range measurements only.

Proof:

Suppose now that another target G moving with a constant velocity, say V_G, is at the same range as the target of interest T. The square of the range at any time t is

$$r^2(t) = [x_T(t) - x_O(t)]^2 + [y_T(t) - y_O(t)]^2 = [x_G(t) - x_O(t)]^2 + [y_G(t) - y_O(t)]^2$$

$$\Leftrightarrow$$

$$\begin{aligned}&[x_T(0) + t\dot{x}_T - \rho\sin(\omega t + \varphi)]^2 + [y_T(0) + t\dot{y}_T - \rho\cos(\omega t + \varphi)]^2 \\ &= [x_G(0) + t\dot{x}_G - \rho\sin(\omega t + \varphi)]^2 + [y_G(0) + t\dot{y}_G - \rho\cos(\omega t + \varphi)]^2 \\ &\forall t.\end{aligned}$$

or, equivalently,

$$\begin{aligned}&x_T^2(0) + y_T^2(0) + \rho^2 + t^2(\dot{x}_T^2 + \dot{y}_T^2) + 2t[\dot{x}_T x_T(0) + \dot{y}_T y_T(0)] \\ &- 2\rho[x_T(0)\sin(\omega t + \varphi) + y_T(0)\cos(\omega t + \varphi)] \\ &- 2t\rho[\dot{x}_T\sin(\omega t + \varphi) + \dot{y}_T\cos(\omega t + \varphi)] \\ = \;& x_G^2(0) + y_G^2(0) + \rho^2 + t^2(\dot{x}_G^2 + \dot{y}_G^2) + 2t[\dot{x}_G x_G(0) + \dot{y}_G y_G(0)] \\ &- 2\rho[x_G(0)\sin(\omega t + \varphi) + y_G(0)\cos(\omega t + \varphi)] \\ &- 2t\rho[\dot{x}_G\sin(\omega t + \varphi) + \dot{y}_G\cos(\omega t + \varphi)], \forall t.\end{aligned}$$

This implies the following five equalities:

$$\forall t \begin{cases} x_G^2(0) + y_G^2(0) = x_T^2(0) + y_T^2(0) \\ \dot{x}_G^2 + \dot{y}_G^2 = \dot{x}_T^2 + \dot{y}_T^2 \\ \dot{x}_G x_G(0) + \dot{y}_G y_G(0) = \dot{x}_T x_T(0) + \dot{y}_T y_T(0) \\ x_G(0)\sin(\omega t + \varphi) + y_G(0)\cos(\omega t + \varphi) \\ \qquad = x_T(0)\sin(\omega t + \varphi) + y_T(0)\cos(\omega t + \varphi) \\ \dot{x}_G\sin(\omega t + \varphi) + \dot{y}_G\cos(\omega t + \varphi) \\ \qquad = \dot{x}_T\sin(\omega t + \varphi) + \dot{y}_T\cos(\omega t + \varphi). \end{cases}$$

The last two equations are equivalent to

$$\begin{bmatrix}\sin(\omega t + \varphi) \\ \cos(\omega t + \varphi)\end{bmatrix}^{\mathsf{T}} [P_G(0) - P_T(0)] = 0 \quad \forall t \text{ and}$$

$$\begin{bmatrix}\sin(\omega t + \varphi) \\ \cos(\omega t + \varphi)\end{bmatrix}^{\mathsf{T}} [V_G - V_T] = 0 \quad \forall t. \text{ Since } \begin{bmatrix}\sin(\omega t + \varphi) \\ \cos(\omega t + \varphi)\end{bmatrix} \text{ spans}$$

the whole two-dimensional space, $P_G(0) = P_T(0)$ and $V_G = V_T$.

QED.

More generally, if the observer's trajectory contains at least one arc of a circle, then the trajectory of any target having a constant velocity is observable in ROTMA, as in BOTMA [6].

V. Observability when observer is in CA motion

In a first time we propose a necessary and sufficient observability condition. In a second time we propose to define the set of 'ghost targets' when de trajectory of the target is not observable.

A. Necessary and sufficient observability condition

The following analysis will be conducted for the relative motion with respect to the observer's trajectory. The question is to identify vectors $X = [x \quad y \quad \dot{x} \quad \dot{y}]^{\mathsf{T}}$ such that

$$\left\| \begin{bmatrix} x + t\dot{x} \\ y + t\dot{y} \end{bmatrix} - \frac{t^2}{2}\Gamma \right\| = \|P_{OT}(t)\|, \quad \forall t \qquad (1)$$

We can already identify two solutions: $X_{OT} = [x_{OT}(0) \quad y_{OT}(0) \quad \dot{x}_{OT}(0) \quad \dot{y}_{OT}(0)]^{\mathsf{T}}$, and $X_{OG}^S = [x_{OT}(0) \quad -y_{OT}(0) \quad \dot{x}_{OT}(0) \quad -\dot{y}_{OT}(0)]^{\mathsf{T}}$ (since $\gamma_y = 0$).

We will show below that most of the time, other ghost-targets can exist.

Lemma 3
The set of solutions $\{ [x \quad y \quad \dot{x} \quad \dot{y}]^{\mathsf{T}} \}$ *of (1) is defined by the following equations:*

$$x^2 + y^2 = x_{OT}^2(0) + y_{OT}^2(0) \qquad (2)$$

$$x\dot{x} + y\dot{y} = x_{OT}(0)\dot{x}_{OT}(0) + y_{OT}(0)\dot{y}_{OT}(0) \qquad (3)$$

$$\dot{x}^2 + \dot{y}^2 - x\gamma_x = \dot{x}_{OT}^2(0) + \dot{y}_{OT}^2(0) - x_{OT}(0)\gamma_x \qquad (4)$$

$$\dot{x}\gamma_x = \dot{x}_{OT}(0)\gamma_x \qquad (5)$$

or equivalently by

$$x^2 + y^2 = r_0^2 \qquad (6)$$

$$x\dot{x} + y\dot{y} = r_0 v_r \cos(\theta_0 - h_r) \qquad (7)$$

$$\dot{x}^2 + \dot{y}^2 - x\gamma_x = v_r^2\cos^2 h_r - r_0\gamma_x\sin\theta_0 \qquad (8)$$

$$\dot{x}\gamma_x = v_r\gamma_x\sin h_r \qquad (9)$$

Note that (5) implies $\dot{x} = \dot{x}_{OT}(0)$.

The next proposition gives us a guide to find all the solutions.

Proposition 6
The vectors
$X_{OT} = [x_{OT}(0) \quad y_{OT}(0) \quad \dot{x}_{OT}(0) \quad \dot{y}_{OT}(0)]^{\mathsf{T}}$ *and*
$X_{OG}^S = [x_{OT}(0) \quad -y_{OT}(0) \quad \dot{x}_{OT}(0) \quad -\dot{y}_{OT}(0)]^{\mathsf{T}}$ *are solution of (1).*
Any other solution $X_{OG} = [x \quad y \quad \dot{x} \quad \dot{y}]^{\mathsf{T}}$ satisfies

$$x^2 + \frac{v_r^2}{\gamma_x}x + r_0\frac{v_r^2}{\gamma_x}\sin(\theta_0 - 2h_r) - r_0^2 = 0 \text{ and } \dot{x} = \dot{x}_{OT}(0).$$

Proof: We exploit here the four equations of Lemma 3. In order to render the proof lighter, we introduce the following

notations:

$a = r_0^2$, $b = r_0 v_r \cos(\theta_0 - h_r)$, and $c = v_r^2 - r_0 \gamma_x \sin\theta_0$.

From (5), we have $\dot{x} = \dot{x}_{OT}(0)$. Reporting this first result in (8), we get $\dot{x}_{OT}^2(0) + \dot{y}^2 - x\gamma_x = c$. Then, multiplying both sides of this equation by y^2, we get

$$y^2\dot{x}_{OT}^2(0) + y^2\dot{y}^2 - x\,y^2\gamma_x = cy^2. \quad (10)$$

Eq. (7) implies $y^2\dot{y}^2 = [b - x\dot{x}_{OT}(0)]^2$. Inserting this into (10), we get

$$y^2\dot{x}_{OT}^2(0) + (b - x\dot{x})^2 - x\,y^2\gamma_x = cy^2. \quad (11)$$

Finally, using Eq. (6), we replace y^2 by $a - x^2$ in (11), and we end up with the following equation:

$$(a - x^2)\dot{x}_{OT}^2(0) + [b - x\dot{x}_{OT}(0)]^2 - x(a - x^2)\gamma_x = c(a - x^2). \quad (12)$$

The cubic equation (12) has at most three real roots (one of them is $x_{OT}(0)$). Let us denote the three solutions as x_i $(i = 0, 1, 2)$ (in some cases, only one or two roots exist). For convenience, the root $x_{OT}(0)$ will be denoted x_0.

To compute the two other roots, we first develop (12):

$$x^3\gamma_x + x^2[-\dot{x}_{OT}^2(0) + \dot{x}_{OT}^2(0) + c] + x[-2b\,\dot{x}_{OT}(0) - a\gamma_x] + \text{cst} = 0$$

$$\Leftrightarrow \quad x^3\gamma_x + x^2c - x[2b\,\dot{x}_{OT}(0) + a\gamma_x] + \text{cst} = 0. \quad (13)$$

Since x_0 is a root, we have

$$x_0^3\gamma_x + x_0^2c - x_0[2b\,\dot{x}_{OT}(0) + a\gamma_x] + \text{cst} = 0. \quad (14)$$

The difference (13) – (14) is

$$(x^3 - x_0^3)\gamma_x + (x^2 - x_0^2)c - (x - x_0)[2b\,\dot{x}_{OT}(0) + a\gamma_x] = 0,$$

or, equivalently,

$$(x - x_0)\{(x^2 + x_0^2 + x\,x_0)\gamma_x + (x + x_0)c - [2b\,\dot{x}_{OT}(0) + a\gamma_x]\} = 0 \quad (15)$$

Dividing (15) by $(x - x_0)$, we get

$$(x^2 + x_0^2 + x\,x_0)\gamma_x + (x + x_0)c - [2b\,\dot{x}_{OT}(0) + a\gamma_x] = 0 \quad (16),$$

which is a quadratic equation.

Rearranging the terms of (16), we end up with

$$x^2\gamma_x + x(x_0\gamma_x + c) + x_0^2\gamma_x + x_0\,c - [2b\,\dot{x}_{OT}(0) + a\gamma_x] = 0$$

or equivalently

$$x^2 + x\frac{x_0\gamma_x + c}{\gamma_x} + x_0\frac{x_0\gamma_x + c}{\gamma_x} - \left[2\frac{b}{\gamma_x}\dot{x}_{OT}(0) + a\right] = 0.$$

Now, we replace the terms a, b,, and c by their respective values. The equation to be solved is hence

$$x^2 + x\frac{v_r^2}{\gamma_x} + r_0\frac{v_r^2}{\gamma_x}\sin\theta_0 - 2r_0\frac{v_r^2}{\gamma_x}\cos(\theta_0 - h_r)\sin h_r - r_0^2 = 0$$

$$\Leftrightarrow \quad x^2 + x\frac{v_r^2}{\gamma_x} + r_0\frac{v_r^2}{\gamma_x}[\sin\theta_0 - 2\cos(\theta_0 - h_r)\sin h_r] - r_0^2 = 0$$

$$\Leftrightarrow \quad x^2 + x\frac{v_r^2}{\gamma_x} + r_0\frac{v_r^2}{\gamma_x}\sin(\theta_0 - 2h_r) - r_0^2 = 0.$$

QED

This proposition tells us that the number of ghost is finite and the observability must be studied by means of the polynomial function $Q(x) = x^2 + \beta x + r_0\beta S - r_0^2$, with $\beta = \frac{v_r^2}{\gamma_x}$ and $S = \sin(\theta_0 - 2h_r)$. Note that β is negative. By convention, the heading h_r is zeroed when $v_r = 0$.

Now, we are able to give a necessary and sufficient observability condition.

Proposition 7: Observability criterion when observer is in CA motion

Assuming that the observer is traveling with a constant acceleration vector, the trajectory of the target is observable from at least four range measurements acquired at different times if and only if O *and* T *are on an RDVR-I.*

Proof:

Firstly, suppose that the trajectory of the target is observable. Then, $X_{OT} = X_{OG}^S$, which implies $y_{OT}(0) = \dot{y}_{OT}(0) = 0$, that is $P_{OT}(0)$, $V_{OT}(0)$, and Γ are collinear; so $P_{OT}(0) = \eta\Gamma$ and $V_{OT}(0) = \lambda\Gamma$. In terms of coordinates, this yields $x_{OT}(0) = \eta\gamma_x$, $\dot{x}_{OT}(0) = \lambda\gamma_x$, $y_{OT}(0) = 0$, and $\dot{y}_{OT}(0) = 0$. Hence $\beta = \frac{\dot{x}_{OT}^2(0)}{\gamma_x}$,

$r_0 S = r_0\sin(\theta_0 \pm \pi) = -r_0\sin\theta_0 = -x_{OT}(0)$, and consequently

$$Q(x) = x^2 + \frac{\dot{x}_{OT}^2(0)}{\gamma_x}x - x_{OT}(0)\frac{\dot{x}_{OT}^2(0)}{\gamma_x} - x_{OT}^2(0).$$

Formally, the roots of $Q(x) = 0$ are $x = x_{OT}(0)$ and $x = -x_{OT}(0) - \frac{\dot{x}_{OT}^2(0)}{\gamma_x}$. Since the trajectory of the target is observable, either $x_{OT}(0) = -x_{OT}(0) - \frac{\dot{x}_{OT}^2(0)}{\gamma_x}$, in other words, $x_{OT}(0)$ is a double root (this corresponds to $\eta = -\frac{\lambda^2}{2}$), or $-x_{OT}(0) - \frac{\dot{x}_{OT}^2(0)}{\gamma_x}$ is an unacceptable physical solution; that is $\left[x_{OT}(0) + \frac{\dot{x}_{OT}^2(0)}{\gamma_x}\right]^2 > r_0^2$. Because $r_0^2 = x_{OT}^2(0)$, this inequality is equivalent to $\lambda^2 > -2\eta$. The target and the observer are on an RDVR-I from Proposition 2.

Conversely, suppose that the target and the observer are on an RDVR-I; that is $P_{OT}(0) = \eta\Gamma$ and $V_{OT}(0) = \lambda\Gamma$, and $\lambda^2 \geq -2\eta$. We have

$$[x_{OT}(0) \quad y_{OT}(0) \quad \dot{x}_{OT}(0) \quad \dot{y}_{OT}(0)]^{\mathsf{T}} = [\eta\gamma_x \quad 0 \quad \lambda\gamma_x \quad 0]^{\mathsf{T}}.$$

Under this assumption, the scalars *a*, *b*, and *c* (defined in the

proof of Proposition 6) take the following values:
$a=\eta^2\gamma_x^2$, $b=\lambda\eta\gamma_x^2$, $c=\left(\lambda^2-\eta\right)\gamma_x^2$. So, the set of equations (6), (7), and (9) becomes

$$x^2+y^2=\eta^2\gamma_x^2 \quad (17)$$

$$\lambda\gamma_x x+y\dot{y}=\lambda\eta\gamma_x^2 \quad (18)$$

$$\lambda^2\gamma_x^2+\dot{y}^2-x\gamma_x=\left(\lambda^2-\eta\right)\gamma_x^2 \quad (19)$$

which is equivalent to

$$y^2=\eta^2\gamma_x^2-x^2 \quad (20)$$

$$y\dot{y}=\lambda\gamma_x\left(\eta\gamma_x-x\right) \quad (21)$$

$$\dot{y}^2=\gamma_x\left(x-\eta\gamma_x\right) \quad (22)$$

Taking the square of (21), we obtain $y^2\dot{y}^2=\lambda^2\gamma_x^2\left(\eta\gamma_x-x\right)^2$.

Then using (20) and (22), we get $\gamma_x\left(\eta^2\gamma_x^2-x^2\right)\left(x-\eta\gamma_x\right)=\lambda^2\gamma_x^2\left(\eta\gamma_x-x\right)^2$, which can be factorized into $-\left(\eta\gamma_x-x\right)^2\left(\eta\gamma_x+x\right)=\lambda^2\gamma_x\left(\eta\gamma_x-x\right)^2$ (23).

If $x\neq\eta\gamma_x$, then (23) is equivalent to $-\left(\eta\gamma_x+x\right)=\lambda^2\gamma_x$, from which we derive $x=-\left(\eta+\lambda^2\right)\gamma_x$. Reporting this result into (20), we have $y^2=\left[\eta^2-\left(\eta+\lambda^2\right)^2\right]\gamma_x^2$; that is, $y^2=-\lambda^2\left(2\eta+\lambda^2\right)\gamma_x^2$. As a consequence, $\lambda^2\left(2\eta+\lambda^2\right)\leq 0$. Let us consider the two following cases: (i) $\lambda=0$. Then $\eta\geq 0$ and $y=0$; we deduce from (17) that $x=-\eta\gamma_x$. Reporting this value into (22), we get $\dot{y}^2=-\eta\gamma_x^2$. We deduce that $\eta\leq 0$. Hence $\eta=0$, which is incompatible with the fact that $r(t)>0$, for $t\in[0,T]$. (ii) $\lambda\neq 0$. Then $2\eta+\lambda^2\leq 0$, but by assumption, $2\eta+\lambda^2\geq 0$. Hence $2\eta+\lambda^2=0$. We have $y=0$. Now, from (21), we get $x=\eta\gamma_x$, which is impossible. From this discussion, we conclude that the case $x\neq\eta\gamma_x$ must be rejected.

Hence, $x=\eta\gamma_x$ and consequently $y=0$ and $\dot{y}=0$. The trajectory of the target is observable.

QED.

We face the apparent paradox encountered in [2]: the ranks of the FIM's in ROTMA and in BOTMA are equal then less than 3 since the bearings are constant [2]. The trajectory of the target is observable in ROTMA although the FIM is rank-deficient.

B. Unobservable case: construction of the set of ghosts

We start with the following proposition:

Proposition 8: Existence condition of only one ghost in ROTMA for a CA motion

Assume that the observer is in CA motion. One unique ghost exists if and only if O and T are on an RDVR-II. Moreover, its trajectory is defined by X_{OG}^{S}.

Proof:

First, assume that O and T are on an RDVR-II; that is, $x_{OT}(0)+t_c\dot{x}_{OT}(0)-\frac{1}{2}t_c^2\gamma_x=0$ and $y_{OT}(0)+t_c\dot{y}_{OT}(0)=0$.

Any ghost is also on an RDVR-II with the target. Consequently, we have

$x+t_c\dot{x}-\frac{1}{2}t_c^2\gamma_x=0$ and $y+t_c\dot{y}=0$.

We deduce that $x+t_c\dot{x}=x_{OT}(0)+t_c\dot{x}_{OT}(0)$. Since $\dot{x}=\dot{x}_{OT}(0)$, we conclude that $x=x_{OT}(0)$, and $y=-y_{OT}(0)$ and $\dot{y}=-\dot{y}_{OT}(0)$.

Conversely, assume now that one unique ghost exists. We recall that its trajectory is defined by $X=\begin{bmatrix}x & y & \dot{x} & \dot{y}\end{bmatrix}^{\mathrm{T}}$ and that $\dot{x}=\dot{x}_{OT}(0)$.

First of all, let us prove that $x=x_{OT}(0)$. Indeed, if $x\neq x_{OT}(0)$, then $y=\dot{y}=0$ (otherwise, another ghost exists whose trajectory is defined by $X'=\begin{bmatrix}x & -y & \dot{x} & -\dot{y}\end{bmatrix}^{\mathrm{T}}$, which is in contradiction with the unicity of the ghost). At this point, we necessarily have $X=\begin{bmatrix}x & 0 & \dot{x}_{OT}(0) & 0\end{bmatrix}^{\mathrm{T}}$. Suppose that $y_{OT}(0)\neq 0$ or $\dot{y}_{OT}(0)\neq 0$. Then, the vector $\begin{bmatrix}x_{OT}(0) & -y_{OT}(0) & \dot{x}_{OT}(0) & -\dot{y}_{OT}(0)\end{bmatrix}^{\mathrm{T}}$ defines the trajectory of a ghost. Since the ghost is unique, we have $X=\begin{bmatrix}x & 0 & \dot{x}_{OT}(0) & 0\end{bmatrix}^{\mathrm{T}}=\begin{bmatrix}x_{OT}(0) & -y_{OT}(0) & \dot{x}_{OT}(0) & -\dot{y}_{OT}(0)\end{bmatrix}^{\mathrm{T}}$, which is a contradiction. Hence $y_{OT}(0)=\dot{y}_{OT}(0)=0$. To summarize, we have obtained $X_{OT}=\begin{bmatrix}x_{OT}(0) & 0 & \dot{x}_{OT}(0) & 0\end{bmatrix}^{\mathrm{T}}$, and $X=\begin{bmatrix}x & 0 & \dot{x}_{OT}(0) & 0\end{bmatrix}^{\mathrm{T}}$. Consequently, $x=x_{OT}(0)$, which is in contradiction with the assumption. Hence, $x=x_{OT}(0)$. Therefore, $Q\left(x_{OT}(0)\right)=0$. Let us develop the expression of $Q\left(x_{OT}(0)\right)$:

$Q\left(x_{OT}(0)\right)=x_{OT}^2(0)+\beta\,x_{OT}(0)+r_0\,\beta\,S-r_0^2$, with $\beta=\frac{v_r^2}{\gamma_x}$ and

$S=\sin\left(\theta_0-2h_r\right)=\sin\theta_0\left(\cos^2 2h_r-\sin^2 2h_r\right)-2\cos\theta_0\sin h_r\cos h_r$.

$$\begin{aligned}Q\left(x_{OT}(0)\right)&=-y_{OT}^2(0)+r_0\sin\theta_0\frac{v_r^2}{\gamma_x}\\&\quad+r_0\frac{v_r^2}{\gamma_x}\left[\sin\theta_0\left(\cos^2 2h_r-\sin^2 2h_r\right)-2\cos\theta_0\sin h_r\cos h_r\right]\\&=-y_{OT}^2(0)+r_0\sin\theta_0\frac{v_r^2}{\gamma_x}\left(1+\cos^2 2h_r-\sin^2 2h_r\right)\\&\quad-2r_0\frac{v_r^2}{\gamma_x}\cos\theta_0\sin h_r\cos h_r\\&=-y_{OT}^2(0)+2r_0\sin\theta_0\cos^2 2h_r\frac{v_r^2}{\gamma_x}-2r_0\frac{v_r^2}{\gamma_x}\cos\theta_0\sin h_r\cos h_r\\&=-y_{OT}^2(0)+2\frac{x_{OT}(0)}{\gamma_x}\dot{y}_{OT}^2(0)-2\frac{y_{OT}(0)}{\gamma_x}\dot{x}_{OT}(0)\dot{y}_{OT}(0)\end{aligned}$$

$$Q(x_{OT}(0))=0 \Leftrightarrow \gamma_x = 2\frac{\dot{y}_{OT}(0)}{y_{OT}^2(0)}\left[x_{OT}(0)\dot{y}_{OT}(0)-\dot{x}_{OT}(0)y_{OT}(0)\right].$$

According to Proposition 3, the target and the observer are on an RDVR-II.

QED.

We plan now to identify all the ghosts; for that, we need a tool given by the following lemma.

Lemma 4

The equation $x^2+\beta x+r_0\beta S-r_0^2=0$ *has two real roots defined by* $x_1=\frac{-\beta-\sqrt{\Delta}}{2}$ *and* $x_2=\frac{-\beta+\sqrt{\Delta}}{2}$, *with* $\Delta=(\beta-2r_0S)^2+4r_0^2(1-S^2)$. *Moreover,* $-r_0 \le x_1 \le r_0 \le x_2$.

Lemma 4 will conduct our study according to the following table:

Table I: Values of (x_1, x_2)

	x_1	x_2
Case 1	$x_1=-r_0$	$x_2=r_0$
Case 2	$-r_0<x_1<r_0$	$x_2=r_0$
Case 3	$x_1=r_0$	$x_2=r_0$
Case 4	$x_1=-r_0$	$x_2>r_0$
Case 5	$-r_0<x_1<r_0$	$x_2>r_0$
Case 6	$x_1=r_0$	$x_2>r_0$

Recall that $x_1+x_2=-\beta$ and $x_1x_2=r_0(\beta S-r_0)$. We deduce that Case 1 is the only case where $v_r=0$.

Case 1: $x_1=-r_0$ and $x_2=r_0$

If $\theta_0 \neq \pm\frac{\pi}{2}$, we end up with three ghosts, whose respective trajectories are defined by $X=\begin{bmatrix}-r_0\\0\\0\\\pm\sqrt{-r_0\gamma_x(1+\sin\theta_0)}\end{bmatrix}$ and X_{OG}^{s}.

If $\theta_0=\frac{\pi}{2}$, we get two ghost-trajectories given by $X=\begin{bmatrix}-r_0\\0\\0\\\pm\sqrt{-2r_0\gamma_x}\end{bmatrix}$.

The target is observable if and only if $\theta_0=-\frac{\pi}{2}$.

Case 2: $-r_0<x_1<r_0$ and $x_2=r_0$

If $\theta_0 \neq \frac{\pi}{2}$, then we have three ghosts:

if $\sin h_r>0$, then

$X=\begin{bmatrix}-\beta-r_0\\\varepsilon\sqrt{-\beta(\beta+2r_0)}\\\dot{x}_{OT}(0)\\\varepsilon\sqrt{-\gamma_x(\beta+2r_0)\sin^2 h_r}\end{bmatrix}$, with $\varepsilon=\pm1$;

else $X=\begin{bmatrix}-\beta-r_0\\\varepsilon\sqrt{-\beta(\beta+2r_0)}\\\dot{x}_{OT}(0)\\-\varepsilon\sqrt{-\gamma_x(\beta+2r_0)\sin^2 h_r}\end{bmatrix}$ and X_{OG}^{s}.

If $\theta_0=\frac{\pi}{2}$ we have two ghosts $X=\begin{bmatrix}r_0\\0\\\dot{x}_{OT}(0)\\0\end{bmatrix}$ and X_{OG}^{s}.

Case 3: $x_1=x_2=r_0$

The target is observable if and only if $\theta_0=\frac{\pi}{2}$.

Otherwise, two ghosts exist, defined by $X=\begin{bmatrix}r_0 & 0 & \dot{x}_{OT}(0) & 0\end{bmatrix}^T$ and X_{OG}^{s}.

Case 4: $x_1=-r_0$ and $x_2>r_0$

The target is observable if and only if $\theta_0=-\frac{\pi}{2}$.

Otherwise, three ghosts exist: $X=\begin{bmatrix}-r_0\\0\\\dot{x}_{OT}(0)\\\pm\sqrt{\gamma_x(\beta-2r_0)\cos^2 h_r}\end{bmatrix}$ and X_{OG}^{s}.

Case 5: $-r_0<x_1<r_0$ and $x_2>r_0$

The target is never observable. At most, three ghosts exist, given by

$X=\begin{bmatrix}x_1\\\varepsilon\sqrt{r_0^2-x_1^2}\\\dot{x}_{OT}(0)\\\varepsilon\frac{r_0v_r\cos(\theta_0-h_r)-x_1\dot{x}_{OT}(0)}{\sqrt{r_0^2-x_1^2}}\end{bmatrix}$, with

$\varepsilon=-1,+1$, and X_{OG}^{s}.

This case includes the scenarios where T and O are on an RDVR- II.

Case 6: $x_1=r_0$ and $x_2>r_0$

The target is observable if and only if $\theta_0=\frac{\pi}{2} \Leftrightarrow h_r=\pm\frac{\pi}{2}$. Else three ghosts exist and their

trajectories are defined by $X = \begin{bmatrix} r_0 \\ 0 \\ \dot{x}_{OT}(0) \\ \pm\sqrt{\gamma_x(\beta + 2r_0)\cos^2 h_r} \end{bmatrix}$ and X^{s}_{OG}.

The following Table II is consistent with Propositions 6, 7 and 8. We have to emphasize upon the particularity of Case 1 and Case 2, when the number of ghosts is equal to 2: we gave as a condition that $\theta_0 = \frac{\pi}{2}$ for the target; but this condition has no reason to be satisfied for one ghost. We must hence extend this condition to the two ghosts by writing: for Cases 1 and 2, if the initial bearing of one ghost (or of the target) is equal to $\frac{\pi}{2}$, then two ghosts exist.

Table II: Summarized results

	Necessary condition to have ghosts			
	0 (Observability case: RDVR-I)	*1*	*2*	*3*
Case 1	$\theta_0 = -\frac{\pi}{2}$	*never*	$\theta_0 = \frac{\pi}{2}$	$\theta_0 \neq \pm\frac{\pi}{2}$
Case 2	*never*	*never*	$\theta_0 = \frac{\pi}{2}$	$\theta_0 \neq \frac{\pi}{2}$
Case 3	$\theta_0 = \frac{\pi}{2}$	*never*	$\theta_0 \neq \frac{\pi}{2}$	*never*
Case 4	$\theta_0 = -\frac{\pi}{2}$	*never*	*never*	$\theta_0 \neq -\frac{\pi}{2}$
Case 5	*never*	*RDVR-II*	*never*	*Not RDVR-II*
Case 6	$\theta_0 = \frac{\pi}{2}$	*never*	*never*	$\theta_0 \neq \frac{\pi}{2}$

C. Examples

In this paragraph, we give one example for each case. The initial position of the observer is $P_O(0) = [0 \quad 0]^T$.

In the next figures, the trajectories of the observer and target are shown by thick lines, whereas those of the ghosts are shown by thin lines. The capital letters ("O" for observer, "T" for target, and "G" for ghost) designate the moving objects.

1) One example of observable case (case 1)

The target starts at $[-5000 \quad 0]^T$ (m) with a velocity of $[20 \quad 2]^T$ (m/s). The initial velocity of the observer is $V_O(0) = [10 \quad 2]^T$ (m/s). Its acceleration is $\Gamma = [-0.0416 \quad 0]^T$ (m/s^2). The observer and the target are on an RDVR-I, since $r(t) - \frac{1}{2}\gamma_x t^2 = r_0 + \dot{r}t$ (Proposition 4 is satisfied). Fig. 1. depicts the scenario which lasts 250 s.

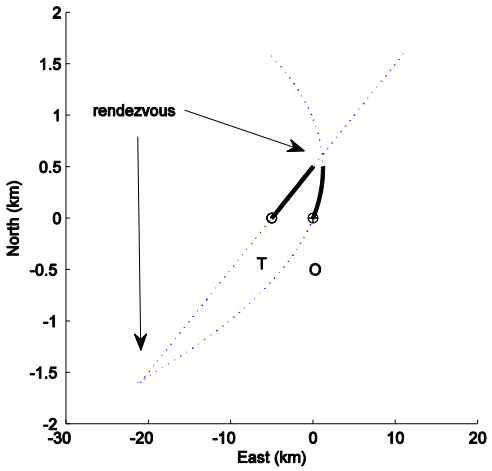


Figure 1. Observable case in ROTMA, RDVR-I.

2) Three examples of unobservable cases (cases 2 & 5)

In the three following examples, the initial observer velocity is the same as previously, and the duration of the scenarios is 360s.

The first example corresponds to case 5 with a RDVR-II.

The target starts at $[3000 \quad 4000]^T$ (m) with a velocity of $[-6 \quad -7]^T$ (m/s). We can readily verify that $\tan\theta(t) = -0.0023t + 0.75$ (see Proposition 4 of [5]). The observer and the target are on a rendezvous route of type II. We can see in Fig. 2. that one ghost exists, and it is also in a rendezvous route with the observer (see Proposition 8).

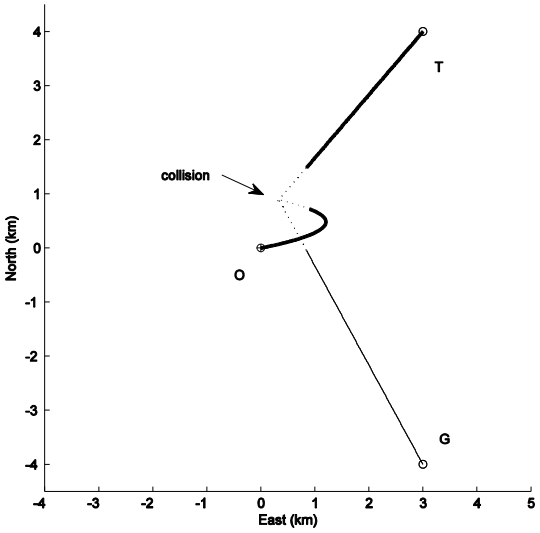


Figure 2. Unobservable case in ROTMA, RDVR-II.

The next example corresponds to case 2

The target starts at $[4000 \;\; 0]^T$ (m) with a velocity of $[25 \;\; 2]^T$ (m/s). We can check that $r(t)+\frac{1}{2}\gamma_x t^2 = 4000+15t$, but the inequality $\dot{r}^2 \geq -2r_0\gamma_x$ is not satisfied since $\dot{r}^2 = 225$ and $-2r_0\gamma_x = 333$. Consequently, the observer and the target are not on a rendezvous route from Proposition 4. The bearings are constant (their value is 90°), two ghosts exist, as announced in Table II. and depicted in Fig. 3. Note that if the role of T and G1 (for example) are inverted (G1 becomes the target and T becomes a ghost), then the bearings of this new target are not constant, but two ghosts only exist.

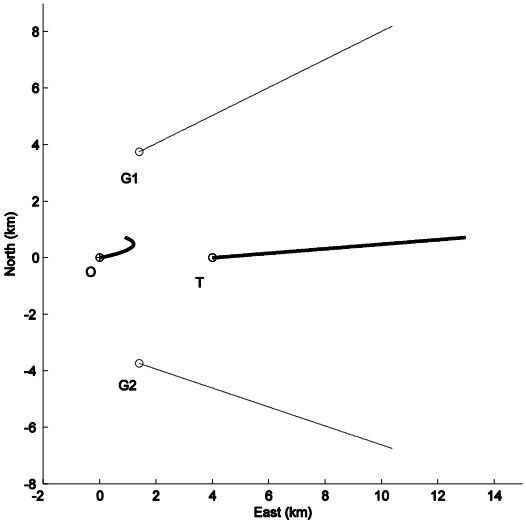


Figure 3. Unobservable case in ROTMA, with constant bearings and no rendezvous route.

In the last example, three ghosts exist. It corresponds to case 5. The target starts at $[2000 \;\; 3464]^T$ (m) with a velocity of $[14.6 \;\; 16.3]^T$ (m/s). The observer and the target are not on a rendezvous route and the bearings are not constant. Three ghosts exist (the maximum number of ghosts), see Fig. 4.

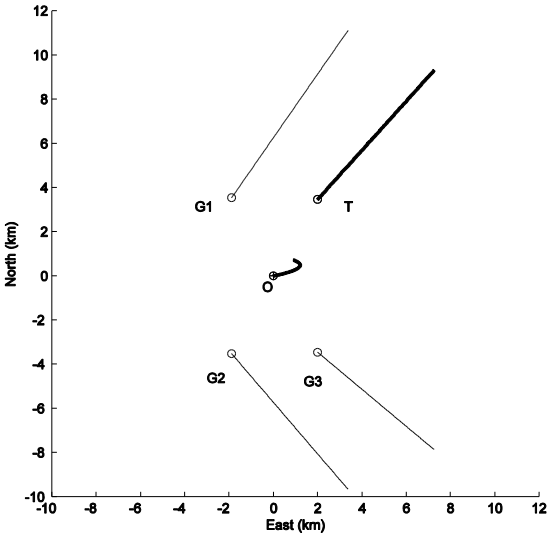


Figure 4. Unobservable case in ROTMA, general case (no rendezvous route and non-constant bearings).

VI. Conclusion

In this paper, observability in ROTMA started in [2] for a target in constant velocity motion have been extended to a smooth observer's maneuver (constant turn motion and constant acceleration motion). We ended up with the following results:

When a part of the displacement of the observer is in an arc of a circle, observability is guaranteed.

When the observer is in constant acceleration motion, the trajectory of the target is observable if, and only if the trajectory of the target relatively to the observer is in a straight line and the target and the observer are in a rendezvous route. In any other scenario, the target is not observable. This demonstrates that even if the observer kinematic is of an order greater than the kinematic of the target, observability is not guaranteed. We proved that in this case, the set of ghost-targets is finite and contains less than three ghost-targets; we gave also the way to construct them from noise-free range measurement. We proposed a range-based observability criterion to know if the target is observable or not.

References

[1] Jauffret, C., and Pillon, D.
Observability in Passive Target Motion Analysis.
IEEE Transactions on Aerospace and Electronic Systems, **32**, 4 (Oct. 1996), 1290–1300.

[2] Pillon, D., Pignol, A.C., and Jauffret, C.
Observability: Range-Only vs. Bearings-Only Target Motion Analysis for a Leg by Leg Observer's Trajectory.
IEEE Transactions on Aerospace and Electronic Systems, **52**, 4 (Aug. 2016), 1667–1678.

[3] Ristic, B., Arulampalam S., and McCarthy J.
Target Motion Analysis Using Range-Only Measurements: Algorithms, Performance and Application to ISAR Data.
Signal Processing, **82** (2002), 273–296.

[4] Sathyan, T., Arulampalam, S., and Mallick, M.,
Multiple Hypothesis Tracking with Multiframe Assignment Using Range and Range-rate Measurements.
In *Proceedings of the International Conference on Information Fusion*, Chicago, USA, July 2011.

[5] Jauffret, C., Pérez, A.C, and, Pillon, D.
Bearings-Only Target Motion Analysis: Observability when the Observer Maneuvers Smoothly, Submitted to the 20th International Conference on Information Fusion, Xi'an, China, Jul. 2017.

[6] Jauffret, C., Pérez, A.C, and, Pillon, D.
Observability: Range-Only vs. Bearings-Only Target Motion Analysis when the Observer Maneuvers smoothly.
In submission in *IEEE Transactions on Aerospace and Electronic Systems*.