# Observability: Range-only vs. Bearings-only Target Motion Analysis for a Leg-by-Leg Observer's Trajectory

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***Abstract*— The aim of so-called range-only target motion analysis (ROTMA) is to estimate the trajectory of a target by a single platform collecting range-only measurements. We focus on the observability analysis when the target is in constant-velocity motion. More precisely, the observability conditions are established when the observer's trajectory is composed of one or several legs. We establish a link between the Fisher information matrix (FIM), and the observability, after having proven the similarity of this matrix with the FIM encountered in bearings-only target motion analysis (BOTMA). We compare observability statuses in BOTMA and ROTMA all along the paper.**

***Index Terms*— Target motion analysis, tracking, range only, bearings only, observability, Fisher information matrix, sonar, radar, electronic support measures.**

## I. Introduction

We consider a unique observer equipped only with a sensor-measuring range. In the case of the presence of a target, estimating its trajectory from the available measurements is called range-only target motion analysis (ROTMA). In this paper, which is a revisited and extended version of [10], we will address the observability problem when the observer has a trajectory composed of one or several legs,[1] and the target is in constant-velocity (CV) motion. The measurements are assumed noise-free.

Even though the formulation of ROTMA is close to that of bearings-only target motion analysis (BOTMA), we will see that these two problems are fundamentally different. This difference will oblige us to conduct the observability analysis in a different way from the analysis in [12]. Moreover, due to a large set of applications, BOTMA has been the source of numerous papers during the last three decades (for example, [11–15, 18]), unlike ROTMA. In short, currently there are fewer applications of ROTMA than of BOTMA.

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[1] A leg is a segment travelled at constant speed.

Among these, we can cite the maritime surveillance radar systems, ISAR (inverse synthetic aperture radar). In [1, 2], the authors consider an observer moving in an arc of a circle, while the target is in CV motion. The range rate is also measured independently of the range. In another maritime application [5], the observer travels along two legs, and the target is in CV motion, which is a typical scenario in BOTMA [14, 15, 22]. We see that for this special case, two solutions coexist, proving that the trajectory of the source is unobservable.

In the robotic domain, ROTMA is also employed, because RF (radio-frequency) sensors must be low-cost, compact, and low-power. In [4], the authors base their analysis upon Stewart's theorem[2] for the target and the observer in CV motion. In [6], the observer makes a succession of arcs of a circle (or an S-shaped trajectory).

For example, following [3], the range measurements can be obtained with a laser. The cases of a target with CV motion, and with constant acceleration motion, are considered: the local observability[3] is analyzed via the Fisher information matrix (FIM). Another given example of ROTMA application can be found for small-size active systems, which provide angle measurements that are biased, very inaccurate, and therefore useless (see [3]).

From the available literature, no general property about global observability has been stated in ROTMA, unlike in BOTMA [11,12]. This is due mainly to the fact that the noiseless system cannot be transformed into an equivalent linear system, to make the observability analysis easier. As a consequence, each type of trajectory (of the observer, as well as the target) has its own particularity. The difficulty is that no general method exists to prove observability in target motion analysis (TMA), when the system of interest is nonlinear; the analysis must be done case by case.

In our paper, the target will be in CV motion (also called one-leg trajectory), whereas the observer will have two different types of trajectories: (i) a one-leg trajectory; and (ii) a two (or three)-leg trajectory. For each type, we will construct the "set

[2] Stewart's theorem establishes properties between the lengths of the sides of a triangle (see https://en.wikipedia.org/wiki/Stewart%27s_theorem or more recently, [19]), without using any angle.

[3] A state vector is said to be locally observable if it is unique in a vicinity. Conversely, a state vector is said to be (globally) observable if it is unique in the whole space.

of solutions", containing at least the target. The other elements of this set (if existing) will be called 'ghosts' (those which are equidistant from the observer all along the scenario) and will show the way to recover their trajectories.

As in any TMA, observability is a fundamental question, because firstly, it allows all the solutions to be identified. Secondly, it allows an adequate state representation to be chosen. The best example is the polar-modified coordinate in BOTMA, since observable and unobservable parts are decoupled [7].

This paper consists of five main sections, followed by the conclusion, appendices, and references.

In Section II, the problem of ROTMA is presented, together with the main notation used in the paper.

In Section III, a convincing (and simple) example is given, which illustrates the fact that analyzing the rank of the Fisher information matrix (FIM) is not a reliable way to study the observability excepted in linear cases. Then, we prove that the FIMs in ROTMA and BOTMA have the same rank, given the two trajectories (of the target and of the observer).

In Section IV, the case where the observer does not maneuver is addressed. The trajectory is not observable, and the set of solutions is uncountable, which is not a surprising result. The interest of this section is to give the equations characterizing this set.

In Section V, the observer travels along two legs with an abrupt change of heading: the set of solutions is proven to contain two elements at most (the "true" one, and the "ghost"). Two equivalent, necessary, and sufficient observability conditions are given. We will end up to a surprising result: for a given scenario, the target is observable in ROTMA if and only if it is not in BOTMA. Then we extend our study to the three-leg case.

The conclusion follows.

In the text, the $d \times d$ identity matrix is denoted as $\mathbf{I}_d$. We will use the symbol $\otimes$ for the Kronecker product of matrices.

## II. Hypotheses, Definitions, and Notations

### A. Kinematics model of target and observer

A target (T) and an observer (O) move in the same plane, given a Cartesian system. The observer measures only the range between the two platforms. The target has CV motion, whereas the observer can maneuver, that is, change its velocity (see Fig. 1). The positions of the observer and the target at time $t \in [0, T]$ are denoted as $P_O(t) = [x_O(t)\ y_O(t)]^\mathsf{T}$ and $P_T(t) = [x_T(t)\ y_T(t)]^\mathsf{T}$, respectively; the corresponding velocity vectors are $V_O(t) = \frac{dP_O(t)}{dt} = [\dot{x}_O(t)\ \dot{y}_O(t)]^\mathsf{T}$ and $V_T = \frac{dP_T(t)}{dt} = [\dot{x}_T\ \dot{y}_T]^\mathsf{T}$.

Since the target has a constant-velocity vector, the equation of its motion is

$$P_T(t) = P_T(0) + t V_T \text{ , or } \begin{cases} x_T(t) = x_T(0) + t\,\dot{x}_T \\ y_T(t) = y_T(0) + t\,\dot{y}_T \end{cases}$$

The motion of the target is, therefore, completely defined (or characterized) by the state vector $X = [x_T(0)\ \ y_T(0)\ \ \dot{x}_T\ \ \dot{y}_T]^\mathsf{T}$. For convenience, we define the relative position vector at time $t$ by $P_{OT}(t) = P_T(t) - P_O(t)$, and the relative velocity vector by $V_{OT}(t) = \frac{dP_{OT}(t)}{dt}$. We assume that at any time the location of the target is different from the location of the observer.

All the angles are clockwise-positive. Subsequently, we will use the symbol $\angle$ to designate angles: for any pair of vectors $\mathsf{U}$ and $\mathsf{W}$, $\angle(\mathsf{U},\mathsf{W})$ is the angle defined by the couple $(\mathsf{U},\mathsf{W})$ referenced to $\mathsf{U}$. When $\mathsf{U}$ is collinear to the northward direction, we will use $\angle\mathsf{W}$ only (for the bearing or heading).
The range and the bearing at time $t$ are given by $R(t) = \| P_{OT}(t) \|$ and $\theta(t) = \angle P_{OT}(t)$, respectively. So, $P_{OT}(t) = R(t) \begin{bmatrix} \sin\theta(t) \\ \cos\theta(t) \end{bmatrix}$; that is, $\begin{cases} x_T(t) = R(t)\sin\theta(t) + x_O(t) \\ y_T(t) = R(t)\cos\theta(t) + y_O(t) \end{cases}$.

So, given the observer's trajectory, the bearing and the range at time t are entirely defined by $X$. We extend the previous notation to emphasize upon this dependence: $\theta(t)$ and $R(t)$ can be denoted $\theta(t, X)$ and $R(t, X)$.

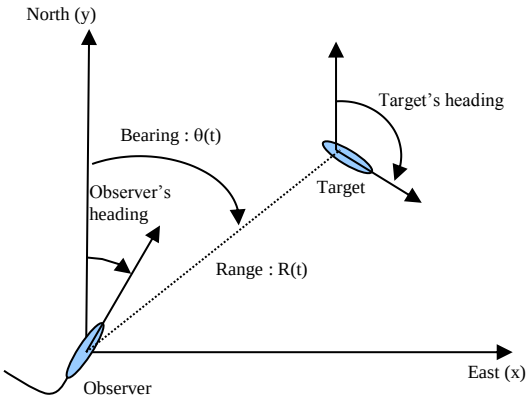



Fig. 1. Typical scenario of TMA.

### B. Measurements model

For the sake of simplicity, the notations $R_k$ and $\theta_k$ will stand for $R(t_k)$ and $\theta(t_k)$, respectively. With no loss of generality, we will assume $t_k = (k-1)\Delta t$. The range collected at $t_k$ and denoted as $R_{m,k}$ obeys the equation $R_{m,k} = R_k + \varepsilon_{R,k}$, where $\varepsilon_{R,k}$ is the measurement noise, which is assumed to be Gaussian and temporally white with zero mean. Its standard

deviation is $\sigma_{R,k}$. The set of $N\ (\geq 4)$ available range measurements is $R_m = \{R_{m,1}, R_{m,2}, \cdots, R_{m,N}\}$.

The aim of ROTMA is to estimate the state vector $X$ given the measurements $R_m$.

Before estimating $X$, the question of observability (i.e. the unicity of $X$) must be posed. As in any problem of TMA, the answer is difficult to establish.

In this paper, the meaning of "observability" is "unicity". The source's trajectory is declared "observable" if the following statements are true:

In BOTMA: $\theta(t, X_G) = \theta(t, X) \Rightarrow X_G = X$,

and in ROTMA: $R(t, X_G) = R(t, X) \Rightarrow X_G = X$.

Another notion of observability exists that is the "local observability": The source's trajectory is declared "locally observable" if the trajectory is observability in a vicinity.

## III. OBSERVABILITY VS. FIM

In the literature dealing with TMA, observability is frequently studied by the rank of the FIM: when the rank of the FIM is equal to the dimension of the state vector, the state vector (or the system) is declared to be observable (see [20] and [21]). This is true when the system is linear, but can lead to wrong conclusions when it is not. In the next subsections, we examine the relationship between the rank of the FIM and observability. We start by a surprising and simple example.

### A. *Simple example of mismatch between regularity of the FIM and Observability*

Let us consider a couple of motionless observers $O_{(1)}$ and $O_{(2)}$, and a motionless target located in $[x_T \quad y_T]^T$. We will consider two situations: in the first, the observers only measure bearings (triangulation); in the second, the observers only measure range (two-range localization).

#### 1) *Triangulation:*

Each observer acquires one bearing ( $\theta_{(i)}$ is acquired by $O_{(i)}$ $i$ =1, 2). Obviously, the point where the two lines of sight cross is the location of the target if it is not on the axis $(O_{(1)}, O_{(2)})$.

Hence, the target is observable if, and only if, it is not on the axis $(O_{(1)}, O_{(2)})$. The FIM is given by

$$\sum_{i=1}^{2} \frac{1}{R_{(i)}^2 \sigma_\theta^2} \begin{bmatrix} \cos^2\theta_{(i)} & -\cos\theta_{(i)}\sin\theta_{(i)} \\ -\cos\theta_{(i)}\sin\theta_{(i)} & \sin^2\theta_{(i)} \end{bmatrix}, \text{ which is of the same rank as } \sum_{i=1}^{2} \begin{bmatrix} \cos^2\theta_{(i)} & -\cos\theta_{(i)}\sin\theta_{(i)} \\ -\cos\theta_{(i)}\sin\theta_{(i)} & \sin^2\theta_{(i)} \end{bmatrix} = \mathbf{F_1}.$$

In this writing, $R_i$ is the range between $O_{(i)}$ and the target.

#### 2) *Two-range localization:*

This time, each sensor measures the range. The set of the potential targets is the intersection of two circles centered on $O_{(i)}$ and of radius $R_{(i)}$ (for $i$ = 1, 2). The target is therefore locally observable but not observable, strictly speaking, unless it is located on the axis $(O_{(1)}, O_{(2)})$. So, the status of observability is the opposite of that in triangulation. The FIM is given by $\sum_{i=1}^{2} \frac{1}{\sigma_R^2} \begin{bmatrix} \sin^2\theta_{(i)} & \cos\theta_{(i)}\sin\theta_{(i)} \\ \cos\theta_{(i)}\sin\theta_{(i)} & \cos^2\theta_{(i)} \end{bmatrix}$, which is of the same rank as $\sum_{i=1}^{2} \begin{bmatrix} \sin^2\theta_{(i)} & \cos\theta_{(i)}\sin\theta_{(i)} \\ \cos\theta_{(i)}\sin\theta_{(i)} & \cos^2\theta_{(i)} \end{bmatrix} = \mathbf{F_2}$.

We can readily verify that $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{F_1} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \mathbf{F_2}$. Hence the matrices $\mathbf{F_1}$ and $\mathbf{F_2}$ have the same rank, but the observability statuses are different in the two cases.

The two situations are illustrated in Figs. 2 and 3.

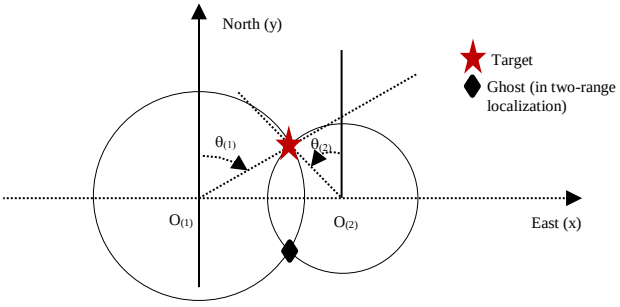



Fig. 2. Observability for triangulation but not for two-range localization.

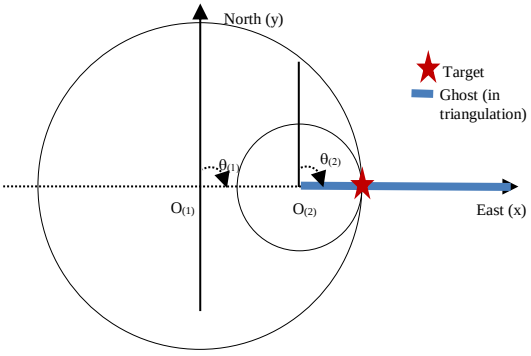



Fig. 3 (a). Observability for two-range localization but not for triangulation.

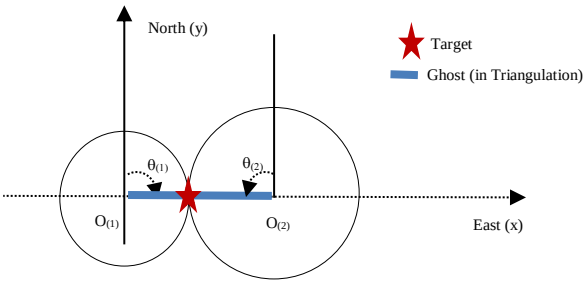
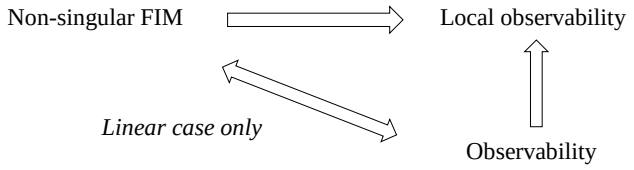



Fig. 3 (b). Observability for two-range localization, but not for triangulation.

*B. Relationship between observability and FIM.*

The relationship of observability, local observability, and regularity of the FIM (under Gaussian assumption) can be illustrated by the following scheme:




The equivalence of the observability and the nonsingularity of the FIM is valid in the Gaussian-linear case: the system is observable if, and only if, the FIM is not singular. With no loss of generality, we prove this when the measurements are scalar: the observability Gramian of the linear system $\begin{cases} X_k = \mathbf{A}_{k,k-1} X_{k-1} \\ Y_k = \mathbf{B}_k X_k \end{cases}$ for $k = 1, \cdots, N$ is $\mathbf{G} = \sum_{k=1}^{N} \mathbf{A}_{k,1}^{\mathsf{T}} \mathbf{B}_k^{\mathsf{T}} \mathbf{B}_k \mathbf{A}_{k,1}$, with $\mathbf{A}_{k,1} = \mathbf{A}_{k,k-1} \mathbf{A}_{k-1,k-2} \cdots \mathbf{A}_{2,1}$ and $\mathbf{A}_{1,1} = \mathbf{I}_d$ (with $\dim(X_k) = d$).

When the measurements are polluted by an additive zero-mean Gaussian noise (its standard deviation being $\sigma_k$), i.e. when we have to deal with the system $\begin{cases} X_k = \mathbf{A}_{k,k-1} X_{k-1} \\ Y_k = \mathbf{B}_k X_k + \varepsilon_k \end{cases}$, the FIM, concerning $X_1$ and given $Y_1, Y_2, \cdots, Y_N$, is $\mathbf{F}(X_1) = \sum_{k=1}^{N} \frac{1}{\sigma_k^2} \mathbf{A}_{k,1}^{\mathsf{T}} \mathbf{B}_k^{\mathsf{T}} \mathbf{B}_k \mathbf{A}_{k,1}$.

Obviously $\mathrm{Rank}\{\mathbf{G}\} = \mathrm{Rank}\{\mathbf{F}(X_1)\}$.

Note that the link between observability and FIM was studied in [16], even when the measurements were polluted by a non-Gaussian noise.

*C. The FIM and observability in BOTMA*

This property can be extended to the problem of BOTMA, whereas the measurement equation is highly nonlinear. The key is that the measurement equation (in noise-free measurements) can be transformed into a linear one: the dynamic (nonlinear system) describing the BOTMA problem is

$$\begin{cases} X_k = \mathbf{\Phi}_{k,k-1} X_{k-1} \\ \theta_k = \tan^{-1}\left[ \dfrac{x_T(t_k) - x_O(t_k)}{y_T(t_k) - y_O(t_k)} \right] \end{cases}$$

where $X_k = [x_T(t_k) \quad y_T(t_k) \quad \dot{x}_T \quad \dot{y}_T]^{\mathsf{T}}$ is the state vector, and $\mathbf{\Phi}_{k,k-1} = \begin{bmatrix} 1 & 0 & t_k - t_{k-1} & 0 \\ 0 & 1 & 0 & t_k - t_{k-1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is the transition matrix.

Using the equality[4] $x_O(t_k)\cos\theta_k - y_O(t_k)\sin\theta_k = x_T(t_k)\cos\theta_k - y_T(t_k)\sin\theta_k$, we transform the preceding system into a linear one:

$$\begin{cases} X_k = \mathbf{\Phi}_{k,k-1} X_{k-1} \\ Y_k = U_k^{\mathsf{T}} X_k \end{cases}$$

with $Y_k = x_O(t_k)\cos\theta_k - y_O(t_k)\sin\theta_k$ and $U_k = [\cos\theta_k \quad -\sin\theta_k \quad 0 \quad 0]^{\mathsf{T}}$, whose observability Gramian is $\mathbf{G} = \sum_{k=1}^{N} \mathbf{\Phi}_{k,1}^{\mathsf{T}} U_k U_k^{\mathsf{T}} \mathbf{\Phi}_{k,1}$.

Let us calculate the FIM. At times $t_k$, $k = 1, \cdots, N$, the observer acquires the bearing measurements $\theta_{m,k} = \theta_k + \varepsilon_{\theta,k}$. The additive noise vector $[\varepsilon_{\theta,1} \quad \cdots \quad \varepsilon_{\theta,N}]^{\mathsf{T}}$ is assumed to be Gaussian with zero mean; its covariance matrix is $diag(\sigma_{\theta,k}^2)$ (known). Under these assumptions, the FIM is $\mathbf{F}_{BOTMA}(X) = \sum_{k=1}^{N} \frac{1}{\sigma_{\theta,k}^2 R_k^2} \mathbf{\Phi}_{k,1}^{\mathsf{T}} U_k U_k^{\mathsf{T}} \mathbf{\Phi}_{k,1}$ (see [14]). Again, $\mathrm{Rank}\{\mathbf{G}\} = \mathrm{Rank}\{\mathbf{F}_{BOTMA}(X)\}$. As a consequence, in BOTMA, observability can be analyzed by computing the rank of the FIM.

*D. The FIM and observability in ROTMA*

Under the above assumptions, a compact expression of the FIM in ROTMA is $\mathbf{F}_{ROTMA}(X) = \sum_{k=1}^{N} \frac{1}{\sigma_{R,k}^2} \mathbf{\Phi}_{k,1}^{\mathsf{T}} W_k W_k^{\mathsf{T}} \mathbf{\Phi}_{k,1}$ with $W_k = [\sin\theta_k \quad \cos\theta_k \quad 0 \quad 0]^{\mathsf{T}}$ and the transition matrix $\mathbf{\Phi}_{k,1} = \begin{bmatrix} 1 & 0 & t_k - t_1 & 0 \\ 0 & 1 & 0 & t_k - t_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

The similarity between $\mathbf{F}_{ROTMA}(X)$ and $\mathbf{F}_{BOTMA}(X)$ is confirmed by the following property:

***Proposition 1***

$$\mathrm{Rank}\{\mathbf{F}_{ROTMA}(X)\} = \mathrm{Rank}\{\mathbf{F}_{BOTMA}(X)\}.$$

Proof:

Let us define the matrix $\mathbf{\Pi} = \mathbf{I}_2 \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. We readily check that $\mathrm{U_k} = \mathbf{\Pi} W_k$ and $\mathbf{\Pi}^{\mathsf{T}} \mathbf{\Phi}_{k,1} = \mathbf{\Phi}_{k,1} \mathbf{\Pi}^{\mathsf{T}}$.

We deduce that $\mathbf{F}_{BOTMA}(X) = \sum_{k=1}^{N} \frac{1}{\sigma_{\theta,k}^2 R_k^2} \mathbf{\Pi} \mathbf{\Phi}_{k,1}^{\mathsf{T}} W_k W_k^{\mathsf{T}} \mathbf{\Phi}_{k,1} \mathbf{\Pi}^{\mathsf{T}}$.

[4] The pseudo-linear estimate and the modified-instrumental method (MIV) in BOTMA are based on this (see [14]).

Since the quantities $\frac{1}{\sigma_{\theta,k}^2 R_k^2}$ are positive,

$$\mathrm{Rank}\{\mathbf{F}_{BOTMA}(X)\} = \mathrm{Rank}\left\{\mathbf{\Pi}\left[\sum_{k=1}^{N}\mathbf{\Phi}_{k,1}^{\mathsf{T}} W_k W_k^{\mathsf{T}}\mathbf{\Phi}_{k,1}\right]\mathbf{\Pi}^{\mathsf{T}}\right\}.$$

Matrix $\mathbf{\Pi}$ being nonsingular, we have

$$\mathrm{Rank}\{\mathbf{F}_{BOTMA}(X)\} = \mathrm{Rank}\left\{\sum_{k=1}^{N}\mathbf{\Phi}_{k,1}^{\mathsf{T}} W_k W_k^{\mathsf{T}}\mathbf{\Phi}_{k,1}\right\}.$$

And for the same reason,

$$\mathrm{Rank}\{\mathbf{F}_{ROTMA}(X)\} = \mathrm{Rank}\left\{\sum_{k=1}^{N}\mathbf{\Phi}_{k,1}^{\mathsf{T}} W_k W_k^{\mathsf{T}}\mathbf{\Phi}_{k,1}\right\}.$$

QED.

Since the ranks of the FIM in ROTMA and of the FIM in BOTMA are equal, we can expect that the conclusion about observability in BOTMA remains valid in ROTMA. As in the given example in subsection III.A, we will prove in the sequel that it is never the case. The fundamental reason for this paradox comes from the fact that a linear transformation of the equation measurement does not exist in ROTMA. Indeed, the state system cannot be transformed into a linear version. This justifies a posteriori our approach, which consists of analyzing observability in ROTMA without using the FIM. Indeed, if an unobservable nonlinear system has a countable set (not a singleton) of solutions, then this system does not have a linear version. The reason is that if such a version exists then the set of solutions is a subspace, hence an uncountable set.

Given a scenario (i.e. the observer's trajectory, and the target's trajectory), our strategy consists of seeking another target (called a "ghost-target," or simply a "ghost," and designated by G) that would be at the same range from the observer as the target of interest, this ghost being in CV motion. Mathematically speaking, we seek $P_G(t)$, the location of G at time t, such that $\|P_G(t)-P_O(t)\| = R(t) = \|P_T(t)-P_O(t)\|$. This equality implies that an orthogonal matrix (rotation or axial symmetry) $\mathbf{H}(t)$ exists, such that $P_G(t)-P_O(t) = \mathbf{H}(t)[P_T(t)-P_O(t)]$. In the coming section, we will prove that $\mathbf{H}(t)$ does not depend on time, when the observer is itself in CV motion.

## IV. THE TRAJECTORY OF THE OBSERVER CONSISTS ONLY OF ONE LEG

In order to make the reading easier, we denote the velocity vector of the observer $V_O(t)$, and the relative velocity vector $V_{OT}(t)$ (which are constant in this case) by $V_O$ and $V_{OT}$, respectively.

In this section, we intend to prove that the orthogonal matrix $\mathbf{H}(t)$ is constant, that is $\mathbf{H}(t) = \mathbf{H}$, then to characterize the set $\mathcal{O}$ of all the ghost-targets located in the same range as the target of interest. Recall that $t_1 = 0$.

### A. Observability analysis

***Proposition 2***

*When the observer and the target are traveling at constant-velocity vectors, the trajectory of the target is not observable. Moreover, the set of ghosts G in CV motion, and at the same distance from the observer as the target, is defined by* $\{P_G(t) = \mathbf{H}[P_T(t)-P_O(t)]+P_O(t),\ \forall t\}$, *where* $\mathbf{H}$ *is an constant orthogonal matrix.*

Proof:

It is obvious that for any constant orthogonal matrix $\mathbf{H}$, the ghost G, whose trajectory is defined by $P_G(t) = \mathbf{H}[P_T(t)-P_O(t)]+P_O(t),\ \forall t$, is in CV motion, and $\|P_{OG}(t)\| = \|P_{OT}(t)\|,\ \forall t$.

Conversely, if G is in CV motion and if $\|P_G(t)\| = \|P_T(t)\|,\ \forall t$, let us prove that a constant orthogonal matrix $\mathbf{H}$ exists such that $P_G(t) = \mathbf{H}[P_T(t)-P_O(t)]+P_O(t),\ \forall t$. To do this, we need to define four vectors, $U_0 = P_{OT}(0)$, $U_1 = V_{OT}$, $W_0 = P_{OG}(0)$, and $W_1 = V_{OG}$, and two matrices : $\mathbf{U} = [U_0 \quad U_1]$ and $\mathbf{W} = [W_0 \quad W_1]$.

$$\|U_0 + tU_1\|^2 = \|W_0 + tW_1\|^2 \qquad (1)$$

$$\Leftrightarrow [1 \quad t]\mathbf{U}^{\mathsf{T}}\mathbf{U}\begin{bmatrix}1\\ t\end{bmatrix} = [1 \quad t]\mathbf{W}^{\mathsf{T}}\mathbf{W}\begin{bmatrix}1\\ t\end{bmatrix}$$

$$\Leftrightarrow \mathbf{U}^{\mathsf{T}}\mathbf{U} = \mathbf{W}^{\mathsf{T}}\mathbf{W} \qquad (2)$$

We deduce that $\mathbf{U}$ and $\mathbf{W}$ have the same rank (1 or 2).

If $\mathrm{Rank}\{\mathbf{U}\} = 1$, then $U_1 = pU_0$ and $W_1 = qW_0$.

(1) $\Leftrightarrow (1+pt)^2\|U_0\|^2 = (1+qt)^2\|W_0\|^2$,

which implies that $\|U_0\| = \|W_0\|$ and $p = q$, and as a consequence, an orthogonal matrix $\mathbf{H}$ exists such that $W_0 = \mathbf{H}U_0$, hence $\mathbf{H}\mathbf{U} = \mathbf{W}$.

If $\mathrm{Rank}\{U\} \neq 1$, then $\mathbf{U}\mathbf{U}^{\mathsf{T}}$ is nonsingular.

$$(2) \Leftrightarrow \mathbf{I}_2 = \mathbf{U}^{-\mathsf{T}}\mathbf{W}^{\mathsf{T}}\mathbf{W}\mathbf{U}^{-1}.$$

We define $\mathbf{H} = \mathbf{W}\mathbf{U}^{-1}$. We note that $\mathbf{H}^{\mathsf{T}} = \mathbf{H}^{-1}$. Hence, $\mathbf{H}$ is an orthogonal matrix and we have $\mathbf{H}\mathbf{U} = \mathbf{W}$.

QED.

### B. Construction of the set of solutions

From Proposition 2, a compact form of $\mathcal{O}$ is

$$\mathcal{O} = \left\{[\mathbf{I}_2 \otimes \mathbf{H}]\begin{bmatrix}P_{OT}(0)\\ V_{OT}\end{bmatrix} + \begin{bmatrix}P_O(0)\\ V_O\end{bmatrix},\right.$$

$$\left.\mathbf{H} \text{ being an orthogonal matrix}\right\}.$$

Any element of $\mathcal{O}$ enables us to construct it entirely. Indeed, let us consider $\begin{bmatrix}P_G(0)\\ V_G\end{bmatrix}$, a particular element of $\mathcal{O}$, where $P_G(0)$ is the initial position of G, and $V_G$ is its velocity.

The set

$$\left\{ \left[\mathbf{I}_2 \otimes \mathbf{H}'\right]\begin{bmatrix} P_{OG}(0) \\ V_{OG} \end{bmatrix}+\begin{bmatrix} P_O(0) \\ V_O \end{bmatrix},\ \mathbf{H}' \text{ being an orthogonal matrix} \right\}$$

is equal to $\mathcal{O}$:

$$\left[\mathbf{I}_2 \otimes \mathbf{H}'\right]\begin{bmatrix} P_{OG}(0) \\ V_{OG} \end{bmatrix}+\begin{bmatrix} P_O(0) \\ V_O \end{bmatrix}=\left[\mathbf{I}_2 \otimes \mathbf{H}'\right]\left[\mathbf{I}_2 \otimes \mathbf{H}\right]\begin{bmatrix} P_{OT}(0) \\ V_{OT} \end{bmatrix}+\begin{bmatrix} P_O(0) \\ V_O \end{bmatrix}$$
$$=\left[\mathbf{I}_2 \otimes \mathbf{H}'\mathbf{H}\right]\begin{bmatrix} P_{OT}(0) \\ V_{OT} \end{bmatrix}+\begin{bmatrix} P_O(0) \\ V_O \end{bmatrix}.$$

The product $\mathbf{H}'\mathbf{H}$ is either a rotation matrix or an axial symmetry matrix.

In practice, we distinguish two cases:

a) either $\mathbf{H}$ is the matrix of a rotation: $\mathbf{H}=\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, where $\alpha$ is the rotation angle; we denote $\mathbf{H}$ as $\mathbf{R}_\alpha$, and we define the set $\mathcal{O}^+=\left\{ \left[\mathbf{I}_2 \otimes \mathbf{R}_\alpha\right]\begin{bmatrix} P_{OT}(0) \\ V_{OT} \end{bmatrix}+\begin{bmatrix} P_O(0) \\ V_O \end{bmatrix},\ \alpha \in [0,\ 2\pi[ \right\}$,

b) or $\mathbf{H}$ is the matrix of axial symmetry relative to the line of the angle $\frac{\beta}{2}$: $\mathbf{H}=\begin{bmatrix} -\cos\beta & \sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$; $\mathbf{H}$ is denoted as $\mathbf{S}_\beta$, and we define the set $\mathcal{O}^-=\left\{ \left[\mathbf{I}_2 \otimes \mathbf{S}_\beta\right]\begin{bmatrix} P_{OT}(0) \\ V_{OT} \end{bmatrix}+\begin{bmatrix} P_O(0) \\ V_O \end{bmatrix},\ \beta \in [0,\ \pi[ \right\}$.

The set of ghost-targets in the same range as the target of interest is $\mathcal{O}=\mathcal{O}^+ \cup \mathcal{O}^-$.

In some circumstances, the set of solutions $\mathcal{O}$ can contain an element corresponding to the motionless mobile ($\|V_G\|=0$). In this case, there exists an orthogonal matrix $\mathbf{H}$, such that $\mathbf{H}(V_T-V_O)+V_O=\vec{0}$. This implies that $\|V_T-V_O\|^2=\|V_O\|^2$.

This equality holds if, and only if, the velocity of the target satisfies the equality $V_T=V_O+\|V_O\|\begin{bmatrix} \sin\psi \\ \cos\psi \end{bmatrix}$, where $\psi$ is a certain angle. For example, if we choose $\mathbf{H}$ as a rotation matrix of a certain angle $\alpha$, that is, $\mathbf{H}=\mathbf{R}_\alpha$, then this angle is given by $\alpha=\angle V_O-\psi+\pi$, since the minimum of the function $\alpha \mapsto \|\mathbf{R}_\alpha(V_T-V_O)+V_O\|^2$ is equal to zero (see Appendix A).

### C. Example

We present the following illustrative scenario.

Starting at $[0 \quad 0]^\mathsf{T}$, the observer travels along the x-axis at a speed equal to 5 m/s. The target of interest starts at $[10{,}000 \quad 15{,}000]^\mathsf{T}$ (m). We construct the velocity of the target such that $V_T=V_O+\|V_O\|\begin{pmatrix} \sin\psi \\ \cos\psi \end{pmatrix}$, with $\psi=-45°$.

In this condition $\|V_T-V_O\|=\|V_O\|$ and the speed of the target is equal to 3.83 m/s and its heading is 22.5°. The total duration is 16min 40s. Figure 4 depicts a subset of $\mathcal{O}$ for $\alpha \in \{0°,\ 5°,\ \cdots,\ 355°\}$ and $\beta \in \{0°,\ 2.5°,\ \cdots,\ 177.5°\}$. In this figure (and in the coming ones), the symbol 'o' shows the initial positions of each trajectory ('O' for the observer, 'T' for the target). For each couple $(\alpha,\ \beta)=\left(\alpha,\ \frac{\alpha}{2}\right)$, two ghost-targets exist, excepted when the ghost-targets are motionless (which is due to this particular scenario).

Note that an assumed knowledge of the initial bearing (see [1]) does not make the trajectory observable in ROTMA, whereas an assumed range makes the trajectory observable in BOTMA (see [17]). This is another asymmetry between ROTMA and BOTMA.

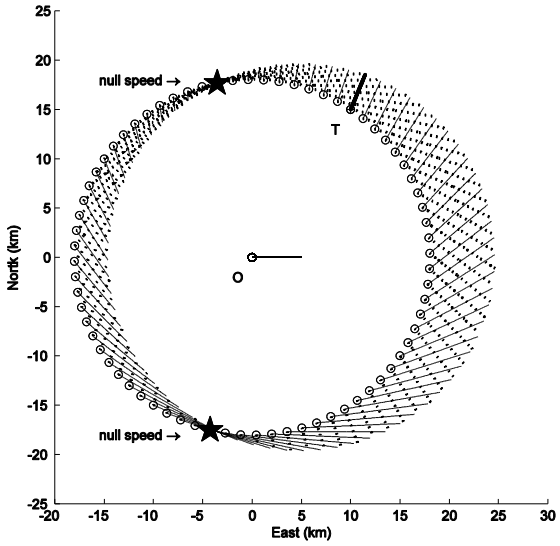



Fig. 4. $\mathcal{O}^+$ is shown by solid lines and $\mathcal{O}^-$ by dashed lines ( "true" target is shown by the bold solid line, and star shows a solution at null speed).

## V. THE TRAJECTORY OF THE OBSERVER CONSISTS OF AT LEAST TWO LEGS

In this section, we examine the case of an observer's trajectory composed of two legs (at least). Let us start with two legs. At time $\tau$, the observer changes its heading: the velocity is $V_{O,1}=\begin{bmatrix} \dot{x}_{O,1} & \dot{y}_{O,1} \end{bmatrix}^\mathsf{T}$ when $t \le \tau$ and $V_{O,2}=\begin{bmatrix} \dot{x}_{O,2} & \dot{y}_{O,2} \end{bmatrix}^\mathsf{T}$ when $t > \tau$ (possibly with $\|V_{O,1}\| \ne \|V_{O,2}\|$), respectively. We end up with the global motion equation $P_O(t)=P_O(\tau)+(t-\tau)V_{O,i}$, with $i=1$, when $t \le \tau$, and $i=2$, when $t > \tau$.

### A. Construction of the set of solutions

The main result concerning the observability is given by the following proposition:

***Proposition 3***

*Let us consider an observer only measuring ranges whose*

*trajectory is composed of two legs (possibly traveled at two different speeds). The* set of solutions *contains two elements at most: the target of interest T, and a ghost-target G, whose trajectory is given by* $P_G(t)=\mathbf{S}_{2\angle(V_{O,2}-V_{O,1})}[P_T(t)-P_O(t)]+P_O(t)$, *where* $\mathbf{S}_{2\angle(V_{O,2}-V_{O,1})}$ *is the matrix of the symmetry around the line spanned by the vector* $V_{O,2}-V_{O,1}$.

Due to its length, the proof of this proposition is given in Appendix B.

Practically, the elements of the matrix $\mathbf{S}_{\angle(V_{O,2}-V_{O,1})}$ can be calculated with the Cartesian coordinates: $\mathbf{S}_{2\angle(V_{O,2}-V_{O,1})}=\frac{1}{\dot{x}_{O,21}^2+\dot{y}_{O,21}^2}\begin{bmatrix}-\dot{y}_{O,21}^2+\dot{x}_{O,21}^2 & 2\dot{x}_{O,21}\dot{y}_{O,21}\\ 2\dot{x}_{O,21}\dot{y}_{O,21} & -\dot{x}_{O,21}^2+\dot{y}_{O,21}^2\end{bmatrix}$ with $\begin{bmatrix}\dot{x}_{O,21}\\ \dot{y}_{O,21}\end{bmatrix}=\begin{bmatrix}\dot{x}_{O,2}-\dot{x}_{O,1}\\ \dot{y}_{O,2}-\dot{y}_{O,1}\end{bmatrix}$, or equivalently, by the polar coordinates: $\mathbf{S}_{2\angle(V_{O,2}-V_{O,1})}=\begin{bmatrix}-\cos 2\gamma & \sin 2\gamma\\ \sin 2\gamma & \cos 2\gamma\end{bmatrix}$ with $\gamma=\angle(V_{O,2}-V_{O,1})$.

This proposition makes it possible to calculate the state vector of the ghost: $X_G=\left[\mathbf{I}_2\otimes\mathbf{S}_{2\angle(V_{O,2}-V_{O,1})}\right]\,[X-X_O]+X_O$, with $X_O=\left[x_O(\tau)\quad y_O(\tau)\quad \dot{x}_{O,i}\quad \dot{y}_{O,i}\right]^{\mathrm{T}}$ ; this computation can be made with $i=1$ or $i=2$, as well.

*B. Necessary and sufficient conditions of observability*

***Proposition 4***

*Let us consider an observer only measuring ranges whose trajectory is composed of two legs (possibly traveled at two different speeds). The trajectory of the target is observable if, and only if, the (nonmeasured) bearings are constant.*

Proof:

The question of observability consists of identifying the conditions under which the equality $P_G(t)=P_T(t)$ holds, $\forall t$. The answer is readily found: $P_G(t)=P_T(t)$ if, and only if, $P_T(t)-P_O(t)$ is the eigenvector of $\mathbf{S}_{2\gamma}$, whatever $t$ is. The vector $P_T(t)-P_O(t)$ is, therefore, on the line spanned by $V_{O,2}-V_{O,1}$: $P_T(t)-P_O(t)=\pm R(t)\begin{bmatrix}\sin\gamma\\ \cos\gamma\end{bmatrix}$; that is, the azimuths are constant ($\theta(t)=\gamma$ or $\theta(t)=\gamma+\pi$).

Conversely, if the azimuths are constant ($\theta(t)=\theta$), then $P_T(t)-P_O(t)=R(t)\begin{bmatrix}\sin\theta\\ \cos\theta\end{bmatrix}$ and, as a consequence, $V_T-V_{O,1}=\dot{R}(t)\begin{bmatrix}\sin\theta\\ \cos\theta\end{bmatrix}$, when $0\le t\le\tau$, and $V_T-V_{O,2}=\dot{R}(t)\begin{bmatrix}\sin\theta\\ \cos\theta\end{bmatrix}$, when $\tau<t\le T$. As $V_T-V_{O,1}$ and $V_T-V_{O,2}$ are two constant vectors, we have $\begin{cases}\dot{R}(t)=\dot{R}_1, \text{ if } 0\le t\le\tau\\ \dot{R}(t)=\dot{R}_2, \text{if } \tau<t\le T\end{cases}$

We deduce that $V_{O,2}-V_{O,1}=\pm(\dot{R}_1-\dot{R}_2)\begin{bmatrix}\sin\theta\\ \cos\theta\end{bmatrix}$. Hence, $P_T(t)-P_O(t)$ and $V_{O,2}-V_{O,1}$ are collinear. As a consequence, $\mathbf{S}_{2\angle(V_{O,2}-V_{O,1})}[P_T(t)-P_O(t)]=P_T(t)-P_O(t)$, that is $P_G(t)=P_T(t)$.

QED

Le Cadre and al. [15] proved that, in BOTMA, when the trajectory of the observer is composed of two legs at constant speed, then the trajectory of the target (whose velocity is constant) is observable if, and only if, the bearings are not constant. Proposition 4 establishes a necessary and sufficient observability condition which is the opposite of the necessary and sufficient observability condition in BOTMA.

***Proposition 5***

*Under the same hypothesis, the trajectory of the target is observable if, and only if, the range rate is constant on* $[0,\tau]$ *and on* $(\tau,T]$.

Proof:

In the proof of Proposition 4, we established that when the bearings are constant, then the range rate is constant during each leg. So, we have now to prove that if the range rate is leg-wise constant, that is $\dot{R}(t)=\dot{R}_1$ if $t\le\tau$, and $\dot{R}(t)=\dot{R}_2$ if $t>\tau$, then the bearings is constant.

As in the proof of Proposition 3, we use an convenient rotation in order to get $P_T(\tau)-P_O(\tau)=\begin{bmatrix}0\\ R(\tau)\end{bmatrix}$. We need to define the angle $c_i=\angle(V_T-V_{O,i})$ for $i=1,2$.

We get on leg *i*:

$P_T(t)-P_O(t)=P_T(\tau)-P_O(\tau)+(t-\tau)(V_T-V_{O,i})$ on leg *i*, or equivalently $P_T(t)-P_O(t)=\begin{bmatrix}0\\ R(\tau)\end{bmatrix}+(t-\tau)\|V_T-V_{O,i}\|\begin{bmatrix}\sin c_i\\ \cos c_i\end{bmatrix}$, which implies that

$$R^2(t)=R^2(\tau)+(t-\tau)^2\|V_T-V_{O,i}\|^2+2(t-\tau)R(\tau)\|V_T-V_{O,i}\|\cos c_i \quad (3)$$

On the other side, since the range rate is constant on each leg, we have $R(t)=R(\tau)+(t-\tau)\dot{R}_i$ on leg *i*. Taking the square of the members of this equation, we get $R^2(t)=R^2(\tau)+(t-\tau)^2\dot{R}_i^2+2(t-\tau)R(\tau)\dot{R}_i$ (4).

Comparing (3) and (4) yields $\cos c_i=\pm1$ ; as a consequence, $\sin c_i=0$. We deduce immediately that $P_T(\tau)-P_O(\tau)$ and $(V_T-V_{O,i})$ are collinear, hence collinear with $P_T(t)-P_O(t)$. The bearings are constant.

QED

### C. Examples

We illustrate our analysis with five scenarios. The first four last 26 minutes and the change of heading is made at 13 minutes. The initial position of the observer is $P_O(0)=[0 \quad 0]^T$. Its heading is equal to 45°, and the speed during this leg is 4 m/s.

The last scenario is drawn from the available literature, and is interesting because it was used in two papers to study several kinds of TMAs (see [5] and [13]).

#### 1) *Scenario 1: constant bearings (observable case).*

During the second leg, the speed and the heading of the observer are equal to 8.49 m/s and –70.53°, respectively. The target starts at $P_T(0)=[4{,}000 \quad 0]^T$ (m), its speed is 5.74 m/s, and its heading is 60.5°, corresponding to velocity $V_T=[5 \quad 2\sqrt{2}]^T$ (m/s). The scenario is presented in Fig. 5.

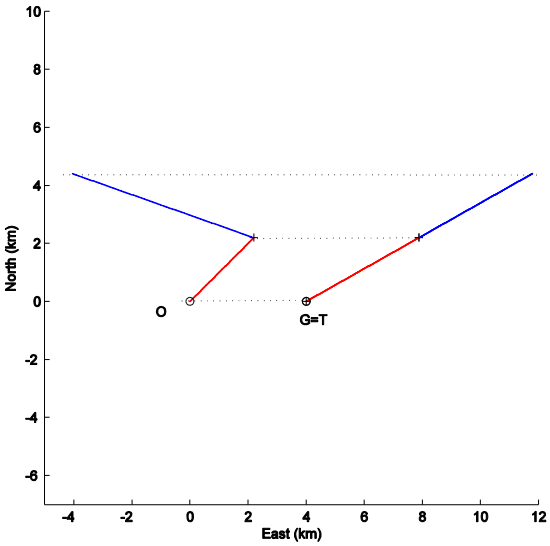



Fig. 5. Observable scenario.

#### 2) *Scenario 2: nonconstant bearings (unobservable case).*

The second speed and the second heading are 5 m/s and 100°. The target starts at $P_T(0)=[1{,}000 \quad 13{,}000]^T$ (m); its velocity is $V_T=[2 \quad -2]^T$ (m/s).

Applying the formula, $P_G(t)=\mathbf{S}_{2\angle(V_{O,2}-V_{O,1})}[P_T(t)-P_O(t)]+P_O(t)$, we compute the initial position of the ghost-target and its velocity: $P_G(0)=[-11{,}668 \quad 5{,}818]^T$ (m), its speed is equal to 7.43 m/s, and its heading is equal to 75.5°.

The whole scenario is depicted in Fig. 6.

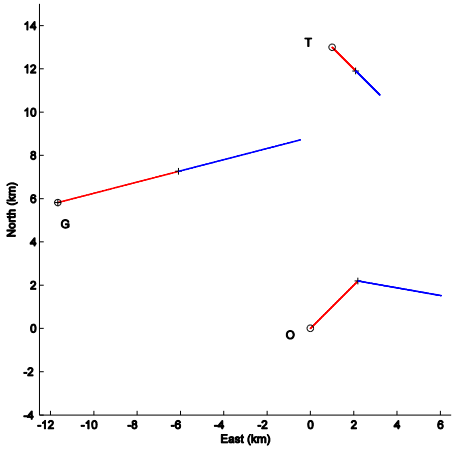



Fig. 6. Unobservable scenario (a target and its ghost).

#### 3) *Scenario 3: constant bearings during the first leg, but not during the second (unobservable case).*

In this scenario, the second heading of the observer is equal to –15°, but it keeps its speed. The target starts from $P_T(0)=[5{,}000 \quad 0]^T$ (m); its speed and heading are 2.83 m/s and 0°, respectively. The corresponding ghost-target is, at the very beginning of the scenario, located at $P_G(0)=[4{,}330 \quad -2{,}500]^T$ (m); it moves with a speed of 4.26 m/s and a heading of 5.12°; its velocity is $V_G=[0.38 \quad 4.24]^T$ (m/s). The trajectories of the three mobiles are depicted in Fig. 7. We insist on the fact that, during the first leg, the range rate and the bearings are constant.

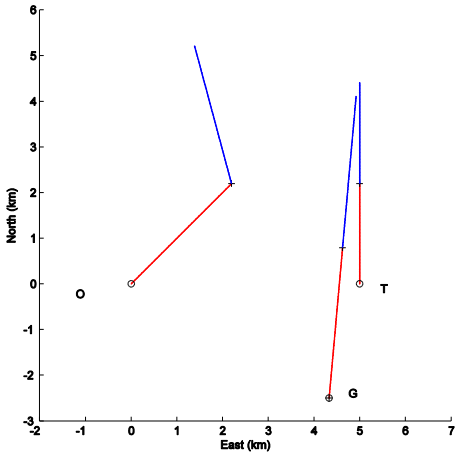



Fig. 7. Bearings are constant during the first leg but not during the second one.

This situation is tricky: the final positions of the two targets are very close, and the speed of the ghost is still 50% higher than the speed of the target.

#### 4) *Scenario 4: colocated target and ghost at a certain time.*

The observer travels on the second leg with a speed equal to 5 m/s, and a new heading equal to –10°. The target starts from $P_T(0)=[-2{,}310 \quad 4{,}489]^T$ (m), with a speed of 2 m/s, and a heading of – 40°. The ghost starts at $P_G(0)=[-5038 \quad -322]^T$ (m), with a speed of 6.77 m/s, and a heading of 13.9°. The trajectories of the three mobiles are depicted in Fig. 8. The respective positions of the target and its ghost are the same at time 880 s (14 min., 40 s).

As a matter of fact, we could construct scenarios for any instant of colocation in $[0, T]$.

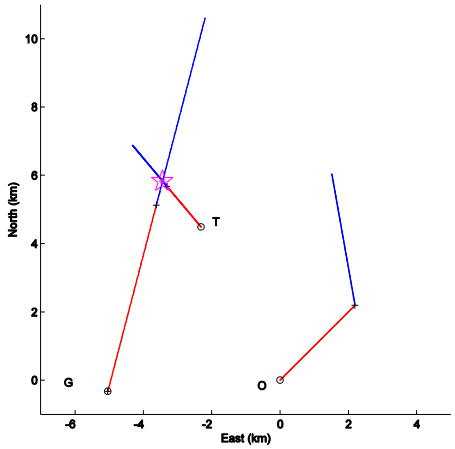



Fig. 8. Target and its ghost are located at the same place (see star).

*5) Scenario 5 used in [5].*

Proposition 5 allows us to deduce the presence of a ghost in the scenario used in [5]. We do not define this scenario, which was introduced originally in a BOTMA study (see [13]). In Fig. 9, the following three trajectories are drawn, of the target (T), ghost (G), and observer (O). The authors of [5] were not concerned by the ghost because they used additional information: an initial bearing measurement in the vicinity of the true one. The trajectories of the three mobiles cross, but not at the same time.

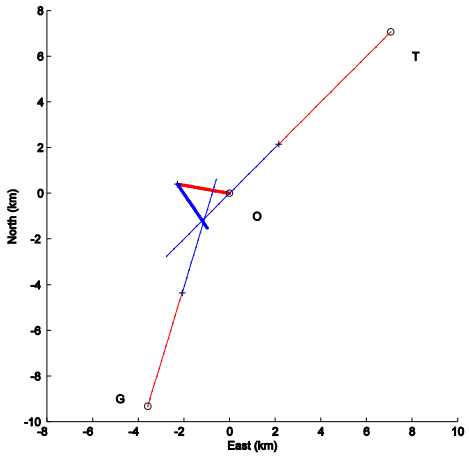



Fig. 9. Target and ghost in the scenario used in [5].

*D. Extension to a trajectory with three legs*

We can extend this result when the observer travels along a third leg: let $V_{O,3}$ be its velocity during this third leg.

***Proposition 6***

*Let us consider an observer only measuring ranges, whose trajectory is composed of three legs. The bearings are not constant during the first two legs. There exists another target denoted $G$, at the same range as the target of interest if, and only if,* $V_{O,3}=(1+\lambda)V_{O,2}-\lambda V_{O,1}$, with $\lambda \neq 0$.

Proof:

Suppose that a ghost G exists. From Proposition 5, we know that during legs 1 and 2 $P_G(t)=\mathbf{S}_{2\angle(V_{O,2}-V_{O,1})}[P_T(t)-P_O(t)]+P_O(t)$ and during legs 2 and 3 $P_G(t)=\mathbf{S}_{2\angle(V_{O,3}-V_{O,2})}[P_T(t)-P_O(t)]+P_O(t)$.

Consequently, during leg 2, we have $\mathbf{S}_{2\angle(V_{O,2}-V_{O,1})}[P_T(t)-P_O(t)]=\mathbf{S}_{2\angle(V_{O,3}-V_{O,2})}[P_T(t)-P_O(t)]$.

The line spanned by $V_{O,2}-V_{O,1}$ is the line spanned by $V_{O,3}-V_{O,2}$. Hence, a nonzero scalar $\lambda$ exists, such that $V_{O,3}-V_{O,2}=\lambda(V_{O,2}-V_{O,1})$.

Conversely, suppose that $V_{O,3}-V_{O,2}=\lambda(V_{O,2}-V_{O,1})$; that is, $V_{O,3}-V_{O,2}$ and $V_{O,2}-V_{O,1}$ are collinear. Then $\mathbf{S}_{2\angle(V_{O,2}-V_{O,1})}=\mathbf{S}_{2\angle(V_{O,3}-V_{O,2})}$. As a consequence, the ghost remains. QED.

Of course, we can add a fourth leg to the observer's trajectory with velocity $V_{O,4}$. If $V_{O,4}-V_{O,3}$ is collinear to $V_{O,3}-V_{O,2}$, the presence of the ghost is maintained. And so on.

Note that if the speeds are equal, that is, $\|V_{O,1}\|=\|V_{O,2}\|=\|V_{O,3}\|=\ldots$, this condition yields $V_{O,3}=V_{O,1}$, $V_{O,4}=V_{O,2}$, and so on. Hence, the trajectory of the target is unobservable if the observer zigzags at constant speed, and the bearings are not constant (again, unlike BOTMA, where the zigzag maneuver can be optimal, under some conditions [14]). Figure 10 gives an example of such a scenario (the trajectories of the zigzagging observer, the target, and the ghost are plotted).

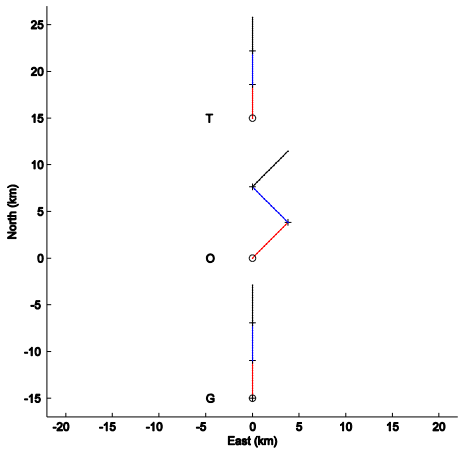



Fig. 10. A zigzagging observer, a target, and its ghost.

As a consequence, knowing the first two velocities of the observer, it is easy to choose the third velocity to get observability.

## VI. Conclusion

In this paper, the properties of range-only target motion analysis were examined in depth, in terms of observability. The target was assumed to be traveling in CV motion. Two types of kinematics of observers were studied: when the observer is in CV motion itself and when the observer's trajectory is composed of two legs. We extended our study to three legs. In the first type, the target's trajectory is unobservable: the set of solutions is uncountable and is defined by a set of orthogonal transformations (rotation or axial symmetry of target's trajectory). In the second type, the trajectory of the target is observable if and only if the range rate is constant, or equivalently if and only if the bearings are constant. Otherwise, this set contains exactly two elements (the true target and a ghost). Finally, augmenting the number of legs will warranty observability, excepted for very special cases.

Moreover, we proved that whatever the scenario, the FIM is similar to the one computed when the range measurements are replaced by bearings measurements (i.e. BOTMA). This provided the opportunity to exhibit an example of nonequivalence between observability and regularity of the FIM: when the observer's trajectory is composed of two legs, the source's trajectory is observable in ROTMA if, and only if, the bearings are constant, whereas the FIM is singular, unlike in the BOTMA (and the converse). This discordancy between observability and regularity of the FIM can occur in other types of TMA, such as the one based on time delay of arrival measurements between two fixed sensors [7]. This is why analyzing observability via the FIM can lead to wrong conclusions, when the state system is nonlinear, or cannot be transformed in a linear form. The overall conclusion is that, for nonlinear systems, observability is very hard to be proved (in some problems, the task could be impossible) whereas local observability is easily obtained by studying the rank of the FIM. But, local observability is useless.

This study could be extended in several directions: firstly, the observability when the observer has a constant acceleration vector, or when it travels in an arc of a circle, must be investigated, together with the extension to 3D space. Secondly, it is known (see [8] and [9]) that in BOTMA, when the observer does not maneuver, the target is observable in special situations; for example, when the target is traveling in an arc of a circle, or according to a two-leg motion model at constant speed. It is legitimate to wonder whether this conclusion in BOTMA remains valid in ROTMA; in particular, is the observer's maneuver unnecessary? The observability of a target whose trajectory is polynomial must be studied, as was done in BOTMA [12]. In a future paper, the authors will address some of these questions, in order to anticipate estimation problems (algorithms, and performance in clean and cluttered environment).

## Appendix

### *Appendix A: A lemma about rotation*

***Lemma***

*Let two bidimensional vectors* $\mathsf{U}$ *and* $\mathsf{W}$*, and their respective angles, be* $u=\angle\mathsf{U}$ *and* $w=\angle\mathsf{W}$*. Then the angle* $\psi=w-u+\pi$ *minimizes the criterion* $\left\|\mathbf{R}_\psi\mathsf{U}+\mathsf{W}\right\|^2$.

Proof:

$$\left\|\mathbf{R}_\psi\mathsf{U}+\mathsf{W}\right\|^2=\left\|\mathbf{R}_\psi\mathsf{U}\right\|^2+\left\|\mathsf{W}\right\|^2+2\mathsf{W}^{\mathsf{T}}\mathbf{R}_\psi\mathsf{U}=\left\|\mathsf{U}\right\|^2+\left\|\mathsf{W}\right\|^2+2\mathsf{W}^{\mathsf{T}}\mathbf{R}_\psi\mathsf{U}$$

.

So, minimizing $\left\|\mathbf{R}_\psi\mathsf{U}+\mathsf{W}\right\|^2$ is equivalent to minimizing $\mathsf{W}^{\mathsf{T}}\mathbf{R}_\psi\mathsf{U}$, and hence to rendering $\mathsf{W}^{\mathsf{T}}\mathbf{R}_\psi\mathsf{U}$ as negative as possible: $\mathbf{R}_\psi\mathsf{U}$ and $\mathsf{W}$ must be collinear and opposite; that is, $\angle\left(\mathbf{R}_\psi\mathsf{U}\right)=w+\pi$.

QED.

### *Appendix B: Proof of Proposition 3 (of section V)*

We proved in Section IV that, when the observer is in CV motion, the set of solutions is the set of images of the true trajectory by any ad-hoc orthogonal matrix. Here, we use this property to analyze the observability when the observer is in CV motion with two different velocities.

Let $\mathbf{H}_1$ and $\mathbf{H}_2$ be the orthogonal matrices for legs 1 and 2.

Given $P_T(t)$, $0\le t\le T$, we plan to solve the two following equations (whose unknowns are $P_G(t)$, $\mathbf{H}_1$ and $\mathbf{H}_2$):

$$\mathbf{H}_1\left[P_T(t)-P_O(t)\right]+P_O(t)=P_G(t), \text{ for leg \#1} \qquad \text{(A-1.a)}$$

and $\mathbf{H}_2\left[P_T(t)-P_O(t)\right]+P_O(t)=P_G(t)$, for leg #2. (A-1.b)

We deduce (by differentiation) that

$$\mathbf{H}_1\left[V_T-V_{O,1}\right]+V_{O,1}=V_G, \text{ for leg \#1} \qquad \text{(A-2.a)}$$

and $\mathbf{H}_2\left[V_T-V_{O,2}\right]+V_{O,2}=V_G$, for leg #2. (A-2.b)

This imposes two constraints: the equality of positions at time $\tau$,

$$\mathbf{H}_1\left[P_T(\tau)-P_O(\tau)\right]=\mathbf{H}_2\left[P_T(\tau)-P_O(\tau)\right] \quad \text{(A-3)}$$

and the equality of velocities,

$$\mathbf{H}_1\left(V_T-V_{O,1}\right)+V_{O,1}=\mathbf{H}_2\left(V_T-V_{O,2}\right)+V_{O,2}. \quad \text{(A-4)}$$

In order to simplify the reading of the coming expressions, we define $\gamma=\angle\left(V_{O,2}-V_{O,1}\right)$, so

$$V_{O,2}-V_{O,1}=\left\|V_{O,2}-V_{O,1}\right\|\begin{bmatrix}\sin\gamma\\ \cos\gamma\end{bmatrix}=\begin{bmatrix}\dot{x}_{O,2}-\dot{x}_{O,1}\\ \dot{y}_{O,2}-\dot{y}_{O,1}\end{bmatrix}. \quad \text{(A-5)}$$

Equation (A-3) is equivalent to

$$\mathbf{H}_1^{-1}\mathbf{H}_2\left[P_T(\tau)-P_O(\tau)\right]=P_T(\tau)-P_O(\tau).$$

The non-null vector $P_T(\tau)-P_O(\tau)$ is hence an eigenvector of the matrix $\mathbf{H}_1^{-1}\mathbf{H}_2$ associated with the eigenvalue 1. The isometries having 1 as an eigenvalue are the identity (see the first case below), and the axial symmetry around the line defined by the vector $P_T(\tau)-P_O(\tau)$ (see the second case).

First case: $\mathbf{H}_1^{-1}\mathbf{H}_2=\mathbf{I}_2$; that is, $\mathbf{H}_1=\mathbf{H}_2=\mathbf{H}$

Equation (A-4) implies that $\mathbf{H}\left(V_{O,2}-V_{O,1}\right)=V_{O,2}-V_{O,1}$. Anew, we deduce from this that $V_{O,2}-V_{O,1}$ is the eigenvector of $\mathbf{H}$ associated with the eigenvalue 1. Hence $\mathbf{H}$ is either the identity, or the matrix of the axial symmetry around the line spanned by $V_{O,2}-V_{O,1}$; that is, using (A-5), $\mathbf{S}_{2\gamma}=\begin{bmatrix}-\cos 2\gamma & \sin 2\gamma\\ \sin 2\gamma & \cos 2\gamma\end{bmatrix}$.

If $\mathbf{H}=\mathbf{I}_2$, then, from (A-1) and (A-2), $\begin{bmatrix}P_G(\tau)\\ V_G\end{bmatrix}=\begin{bmatrix}P_T(\tau)\\ V_T\end{bmatrix}$.

Hence, $P_G(t)=P_T(t),\ \forall t$.

If $\mathbf{H}=\mathbf{S}_{2\gamma}$, then, from (A-1) and (A-2), $\left[\mathbf{I}_2\otimes\mathbf{S}_{2\gamma}\right]\begin{bmatrix}P_T(\tau)-P_O(\tau)\\ V_T-V_{O,i}\end{bmatrix}+\begin{bmatrix}P_O(\tau)\\ V_{O,i}\end{bmatrix}=\begin{bmatrix}P_G(\tau)\\ V_G\end{bmatrix}$ for $i=1,2$.

We deduce that

$$P_G(t)=\mathbf{S}_{2\gamma}\left[P_T(t)-P_O(t)\right]+P_O(t),\ \forall t. \quad \text{(A-6)}$$

Second case: $\mathbf{H}_1^{-1}\mathbf{H}_2$ is the matrix of an axial symmetry.

We have no choice: one matrix is the matrix of a rotation, and the other matrix is one of an axial symmetry. For the sake of simplicity, we will only study the case where $\mathbf{H}_1$ is a rotation matrix (denoted $\mathbf{R}_\alpha$), and $\mathbf{H}_2$ is an axial symmetry matrix (denoted $\mathbf{S}_\beta$), with

$$\mathbf{H}_1=\mathbf{R}_\alpha=\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix} \text{ and } \mathbf{H}_2=\mathbf{S}_\beta=\begin{bmatrix}-\cos\beta & \sin\beta\\ \sin\beta & \cos\beta\end{bmatrix}.$$

Note that $\mathbf{S}_\beta=\mathbf{R}_\beta\mathbf{S}_0$, $\mathbf{R}_\alpha^{-1}=\mathbf{R}_\alpha^{\mathrm{T}}$, and $\mathbf{S}_\alpha^{-1}=\mathbf{S}_\alpha$.

Equations (A-3) and (A-4) are, in this case, rewritten as follows

$$\mathbf{R}_\alpha\left[P_T(\tau)-P_O(\tau)\right]=\mathbf{R}_\beta\mathbf{S}_0\left[P_T(\tau)-P_O(\tau)\right] \quad \text{(A-7)},$$

$$\mathbf{R}_\alpha\left(V_T-V_{O,1}\right)+V_{O,1}=\mathbf{R}_\beta\mathbf{S}_0\left(V_T-V_{O,2}\right)+V_{O,2} \quad \text{(A-8)}.$$

Using the ad-hoc rotation, we turn the whole scenario, in order to get $P_T(\tau)-P_O(\tau)=\begin{bmatrix}0\\ R(\tau)\end{bmatrix}$, so $x_T(\tau)=x_O(\tau)$, which satisfies $\mathbf{S}_0\left[P_T(\tau)-P_O(\tau)\right]=P_T(\tau)-P_O(\tau)$. Reporting this equality in (A-7) implies that $\alpha=\beta$. Consequently, equation (A-8) is now

$$\mathbf{R}_\alpha\left(V_T-V_{O,1}\right)+V_{O,1}=\mathbf{R}_\alpha\mathbf{S}_0\left(V_T-V_{O,2}\right)+V_{O,2}$$

$$\Leftrightarrow\ \mathbf{R}_\alpha\left(V_T-V_{O,1}\right)-\mathbf{R}_\alpha\mathbf{S}_0\left(V_T-V_{O,2}\right)=V_{O,2}-V_{O,1}. \quad \text{(A-9)}$$

Now, multiplying the terms (A-9) by the matrix $\left(\mathbf{I}_2+\mathbf{S}_0\right)\mathbf{R}_\alpha^{\mathrm{T}}$, we readily obtain

$$\left(\mathbf{I}_2+\mathbf{S}_0\right)\left(V_{O,2}-V_{O,1}\right)=\left(\mathbf{I}_2+\mathbf{S}_0\right)\mathbf{R}_\alpha^{\mathrm{T}}\left(V_{O,2}-V_{O,1}\right),$$

$$\Leftrightarrow\ \left(\mathbf{I}_2+\mathbf{S}_0\right)\left(\mathbf{I}_2-\mathbf{R}_\alpha^{\mathrm{T}}\right)\left(V_{O,2}-V_{O,1}\right)=0.$$

$$\Leftrightarrow\ \begin{bmatrix}0 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}1-\cos\alpha & \sin\alpha\\ -\sin\alpha & 1-\cos\alpha\end{bmatrix}\begin{bmatrix}\sin\gamma\\ \cos\gamma\end{bmatrix}=0$$

which leads to the equality $\cos(\alpha-\gamma)=\cos\gamma$. This equality holds if $\alpha=0$ or if $\alpha=2\gamma$.

If $\alpha=0$ (which implies $\mathbf{H}_1=\mathbf{R}_0=\mathbf{I}_2$):

Equation (A-1.a), which is valid for $t\le\tau$ implies that $P_G(t)=P_T(t)$ during the first leg; since T and G move in CV motion, $P_G(t)=P_T(t)$ during the whole scenario.

Now, if $\alpha=2\gamma$ ($\mathbf{H}_1=\mathbf{R}_{2\gamma}$ and $\mathbf{H}_2=\mathbf{S}_{2\gamma}$):

Equation (A-9) is now

$$\mathbf{R}_{2\gamma}\left(V_T-V_{O,1}\right)+V_{O,1}=\mathbf{R}_{2\gamma}\mathbf{S}_0\left(V_T-V_{O,2}\right)+V_{O,2}$$

$$\Leftrightarrow\ V_T-V_{O,1}+\mathbf{R}_{2\gamma}^{\mathrm{T}}V_{O,1}=\mathbf{S}_0\left(V_T-V_{O,2}\right)+\mathbf{R}_{2\gamma}^{\mathrm{T}}V_{O,2}$$

$$\Leftrightarrow\ \left(\mathbf{I}_2-\mathbf{S}_0\right)V_T=V_{O,1}-\mathbf{S}_0V_{O,2}+\mathbf{R}_{2\gamma}^{\mathrm{T}}\left(V_{O,2}-V_{O,1}\right)$$

$$\Leftrightarrow\ \begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\dot{x}_T\\ \dot{y}_T\end{bmatrix}=\begin{bmatrix}\dot{x}_{O,1}\\ \dot{y}_{O,1}\end{bmatrix}-\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\dot{x}_{O,2}\\ \dot{y}_{O,2}\end{bmatrix}+\left\|V_{O,2}-V_{O,1}\right\|\begin{bmatrix}\cos 2\gamma & -\sin 2\gamma\\ \sin 2\gamma & \cos 2\gamma\end{bmatrix}\begin{bmatrix}\sin\gamma\\ \cos\gamma\end{bmatrix}$$

$$\Leftrightarrow\begin{cases}2\dot{x}_T=\dot{x}_{O,1}+\dot{x}_{O,2}-\left\|V_{O,2}-V_{O,1}\right\|\sin\gamma\\ 0=\dot{y}_{O,1}-\dot{y}_{O,2}+\left\|V_{O,2}-V_{O,1}\right\|\cos\gamma.\end{cases}$$

Let us recall that $\left\|V_{O,2}-V_{O,1}\right\|\sin\gamma=\dot{x}_{O,2}-\dot{x}_{O,1}$ and $\left\|V_{O,2}-V_{O,1}\right\|\cos\gamma=\dot{y}_{O,2}-\dot{y}_{O,1}$ - see (A-5) -. The second equation contains no additional information, whereas the first one is equivalent to $2\dot{x}_T=\dot{x}_{O,1}+\dot{x}_{O,2}-\dot{x}_{O,2}+\dot{x}_{O,1}$; that is, $\dot{x}_T=\dot{x}_{O,1}$. Consequently, $V_T-V_{O,1}=\begin{bmatrix}0\\ \dot{R}(t)\end{bmatrix}$; hence, $\left|\dot{R}(t)\right|=\left\|V_T-V_{O,1}\right\|=\left|\dot{R}\right|$ and $P_T(t)-P_O(t)=\begin{bmatrix}0\\ R(\tau)+(t-\tau)\dot{R}\end{bmatrix}$ when $t\le\tau$. This implies that

$\mathbf{R}_{2\gamma}\left[P_T(t)-P_O(t)\right]=\mathbf{S}_{2\gamma}\left[P_T(t)-P_O(t)\right]$ if $t\le\tau$.

Finally, the equations (A-1.a) (A-1.b) are equivalent to $\mathbf{S}_{2\gamma}\left[P_T(t)-P_O(t)\right]+P_O(t)=P_G(t)$ during the whole scenario.

QED.

Acknowledgments

The authors would like to thank Jean Barrère (at the university of Toulon), who proved Proposition 2.

The anonymous reviewers made very encouraging and constructive comments, and we would like to thank them here.

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